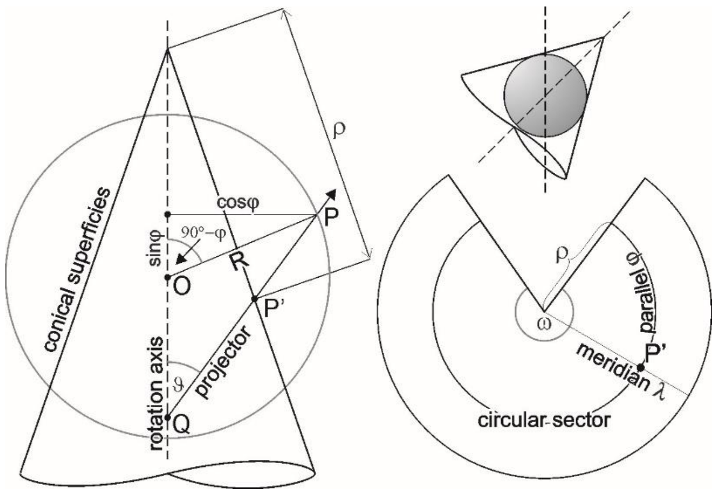
# Lesson_10_B

- Conical map projections of the sphere and the ellipsoid
- Applications of the conformal conical projections

# Conical map projections (1)

Let a central perspective map projection be taken from a spherical surface onto a *superficies of a cone of revolution*. In order to symmetry of the map graticule, the centre has to be on the common rotation axis of the Earth sphere and the cone. Similarly to the considerations in the case of azimuthal and cylindrical projections, after cutting the conical superficies along a generator and unfolded it into plane, it becomes a *circular sector*, the images of the parallels become concentric circular arcs and the images of the meridians become straight lines crossing the common centre.




If the conical superficies is tangent to the sphere then the tangent line (a circle) is the single true scale parallel; in case of cutting superficies, the cutting curves (circles) are the two true scale parallels.

# Conical map projections (2)

Generalizing these properties of the map graticule, the *conical projections* can be defined by their graticule or metagraticule, requiring that

- the images of the parallels are concentric circular arcs
- the images of the meridians are straight lines, converging towards the common center
- the images of the parallels and the meridians are perpendicular to each other
- the angles between the images of the meridians are proportional to the correspondent longitude differences. This proportion denoted by *n,* called *constant of the cone* or *convergency factor*, which equals the ratio of the central angle $\omega$ of the circular sector and the full angle, consequently $0<n<1$.

they are called *normal* (polar) version;

or (in case of spherical Earth) there exists a metacoordinate system, where the above mentioned properties are valid to the metagraticule, and they are called *oblique* or *transverse* (equatorial) version depending on the location of the metapole. In most of the cases the conical projections are applied in normal version, and they have one or two true scale parallels.

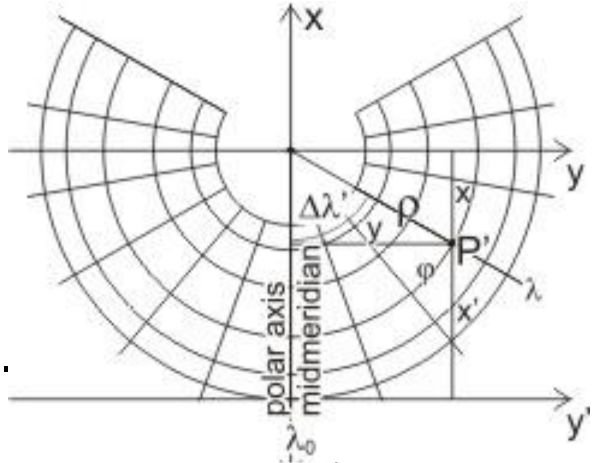
On the map plane, a polar coordinate system can be established having the origin in the common centre of the images of parallels. The polar distance $\rho$ is the radius of parallel on the map, the polar angle denoted by $\Delta\lambda' = (\lambda' - \lambda_0')$. Then $\Delta\lambda'$ gives the angle between the meridians on the map. Due to the proportionality of the angles of the meridians on the map and on the sphere:

$$\Delta\lambda' = n \cdot \Delta\lambda$$

# Conical map projections (3)

The origin of the rectangular and the polar coordinate systems coincide, and the negative part of the axis *x* coincides with the polar axis. Sometimes the axis *y* is shifted to avoid the negative coordinates *x*. Hence the rectangular coordinates of the normal version are ($\Delta\lambda=\lambda-\lambda_0$):

$$x = -\rho(\varphi)\cdot\cos(\Delta\lambda') = -\rho(\varphi)\cdot\cos(n\cdot\Delta\lambda)$$
$$y = \rho(\varphi)\cdot\sin(\Delta\lambda') = \rho(\varphi)\cdot\sin(n\cdot\Delta\lambda)$$




The graticule distortions are

$$h = \frac{\rho(\varphi)\cdot n}{R\cdot\cos\varphi} \qquad k = -\frac{d\rho}{d\varphi}\frac{1}{R}$$

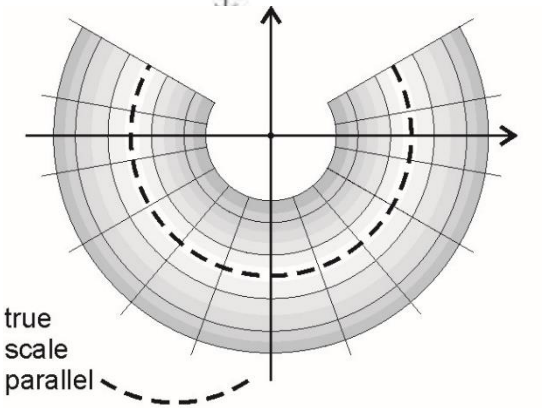
and $\cot\Theta=0$ due to the orthogonality of the graticule.

In the case of transverse or oblique version:

$$x = -\rho(\varphi^*)\cdot\cos(n\cdot\lambda^*)$$
$$y = \rho(\varphi^*)\cdot\sin(n\cdot\lambda^*)$$

**Notable conical projections of the sphere**:

- Lambert *conformal* conical projection
- *Equidistant* conical projection (the meridians and one of the parallels are true scale, aphylactic)
- De l’Isle *equidistant* conical projection (the meridians and two parallels are true scale, aphylactic)
- Albers curved-polar *equal-area* projection (two true scale parallels)




The grayscale level indicates the magnitude of the distortions in the conical projections (*white* shows the negligible distortions)

# Lambert conformal conical projection of the sphere

The equation of the conformity: *h*=*k*, which means that

$$\frac{\rho(\varphi)\cdot n}{R\cdot\cos\varphi}=-\frac{d\rho}{d\varphi}\frac{1}{R}$$

After integration of the separable equation:

$$\ln\rho=n\cdot\ln\left(\frac{1}{\tan\left(45^\circ+\frac{\varphi}{2}\right)}\right)+\ln d$$

(ln*d* is the constant of integration.) The radius $\rho$ of the parallel $\varphi$ on the map:

$$\rho=d\cdot\cot^n\left(45^\circ+\frac{\varphi}{2}\right)=d\cdot\left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)^{\frac{n}{2}}$$

The parameters *d* and *n* can be obtained from the location of the true scale parallel(s). Thus in the case of one true scale parallel $\varphi_s$, the equations

$$n=\sin\varphi_s$$

and

$$d=\frac{R\cdot\cot\varphi_s}{\cot^n\left(45^\circ+\frac{\varphi_s}{2}\right)}$$

give the parameters. Due to $\rho(\varphi=90^\circ)=0$, this projection is pointed-polar, that is the image of the pole is the common centre of the parallels.

# Lambert conformal conical projection of the ellipsoid (1)

The conical projections of the ellipsoid is described by

$$x = -\rho(\Phi)\cdot\cos[n\cdot(\Lambda-\Lambda_0)]$$
$$y = \rho(\Phi)\cdot\sin[n\cdot(\Lambda-\Lambda_0)]$$

The graticule distortions are in this case:

$$h = \frac{\rho(\Phi)\cdot n}{N(\Phi)\cdot\cos\Phi} \qquad k = -\frac{d\rho}{d\Phi}\cdot\frac{1}{M(\Phi)}$$

and $\cot\Theta=0$.

The equation of the conformity is $h=k$, which is in this case

$$\frac{\rho(\Phi)\cdot n}{N(\Phi)\cdot\cos\Phi} = -\frac{d\rho}{d\Phi}\frac{1}{M(\Phi)}$$

Its solution for the function $\rho$ is:

$$\rho = d\cdot\cot^n\left(45^\circ+\frac{\Phi}{2}\right)\cdot\left(\frac{1+e\cdot\sin\Phi}{1-e\cdot\sin\Phi}\right)^{\frac{n\cdot e}{2}} = d\cdot\left(\frac{1-\sin\Phi}{1+\sin\Phi}\right)^{\frac{n}{2}}\cdot\left(\frac{1+e\cdot\sin\Phi}{1-e\cdot\sin\Phi}\right)^{\frac{n\cdot e}{2}}$$

The values of the parameters $n$ and $d$ depend on the location of the true scale parallel(s). In the case of one true scale parallel $\Phi_s$, the equations:

$n=\sin\Phi_s$

and

$$d = \frac{N(\Phi_s)\cdot\cot\Phi_s}{\cot^n\left(45^\circ+\frac{\Phi_s}{2}\right)}\cdot\left(\frac{1-e\cdot\sin\Phi_s}{1+e\cdot\sin\Phi_s}\right)^{\frac{n\cdot e}{2}} = \frac{N(\Phi_s)\cdot\cot\Phi_s}{\left(\frac{1-\sin\Phi_s}{1+\sin\Phi_s}\right)^{\frac{n}{2}}}\cdot\left(\frac{1-e\cdot\sin\Phi_s}{1+e\cdot\sin\Phi_s}\right)^{\frac{n\cdot e}{2}}$$

# Lambert conformal conical projection of the ellipsoid (2)

Supposing that $\Phi_1$ and $\Phi_2$ are the true scale parallels, the following formulae give the parameters *n* and *d*:

$$n = \frac{\ln\left(\dfrac{\cos\Phi_1}{\cos\Phi_2}\cdot\sqrt{\dfrac{1-e^2\cdot\sin^2\Phi_2}{1-e^2\cdot\sin^2\Phi_1}}\right)}{\ln\left(\sqrt{\left(\dfrac{\dfrac{1-\sin\Phi_1}{1+\sin\Phi_1}}{\dfrac{1-\sin\Phi_2}{1+\sin\Phi_2}}\right)\cdot\left(\dfrac{\dfrac{1+e\cdot\sin\Phi_1}{1-e\cdot\sin\Phi_1}}{\dfrac{1+e\cdot\sin\Phi_2}{1-e\cdot\sin\Phi_2}}\right)^e}\right)} = 2\cdot\frac{\ln\cos\Phi_1-\ln\cos\Phi_2+\dfrac{1}{2}\cdot\ln\left(\dfrac{1-e^2\cdot\sin^2\Phi_2}{1-e^2\cdot\sin^2\Phi_1}\right)}{\ln\left(\dfrac{1-\sin\Phi_1}{1+\sin\Phi_1}\right)-\ln\left(\dfrac{1-\sin\Phi_2}{1+\sin\Phi_2}\right)+e\cdot\ln\left(\dfrac{\dfrac{1+e\cdot\sin\Phi_1}{1-e\cdot\sin\Phi_1}}{\dfrac{1+e\cdot\sin\Phi_2}{1-e\cdot\sin\Phi_2}}\right)}$$

and

$$d = \frac{N(\Phi_1)\cdot\cos\Phi_1}{n\cdot\left(\dfrac{1-\sin\Phi_1}{1+\sin\Phi_1}\right)^{\frac{n}{2}}}\cdot\left(\frac{1-e\cdot\sin\Phi_1}{1+e\cdot\sin\Phi_1}\right)^{\frac{n\cdot e}{2}} = \frac{N(\Phi_2)\cdot\cos\Phi_2}{n\cdot\left(\dfrac{1-\sin\Phi_2}{1+\sin\Phi_2}\right)^{\frac{n}{2}}}\cdot\left(\frac{1-e\cdot\sin\Phi_2}{1+e\cdot\sin\Phi_2}\right)^{\frac{n\cdot e}{2}}$$

On occasion, the conformal conical projection with two true scale parallels is called *Lambert-Gauss projection*.

An advantageous property of the conformal conical projection is that the Earth’s orthodromes (geodetic lines) not longer than 3000 km hardly deviate from the straight lines on the map.

# The applications of the conformal conical projections

The World map system „*World Aeronautical Chart*” (WAC, map scale 1 : 1 000 000) of the International Civil Aviation Organization (ICAO) has been using this projection since 1962. Between the 60th parallel south and the 60th parallel north, the reference surface is partitioned into zones of 6° and bands of 4°, so this territory is represented on map sheets including *geographic quadrangles of* $6°\times 4°$. Outside of the parallels of $\pm 60°$, the longitude difference of a geographic quadrangle is greater due to merging two or more zones. The surroundings of the poles are represented in the earlier mentioned ellipsoidal *polar stereographic projection*. There is not any uniform global map coordinate system.

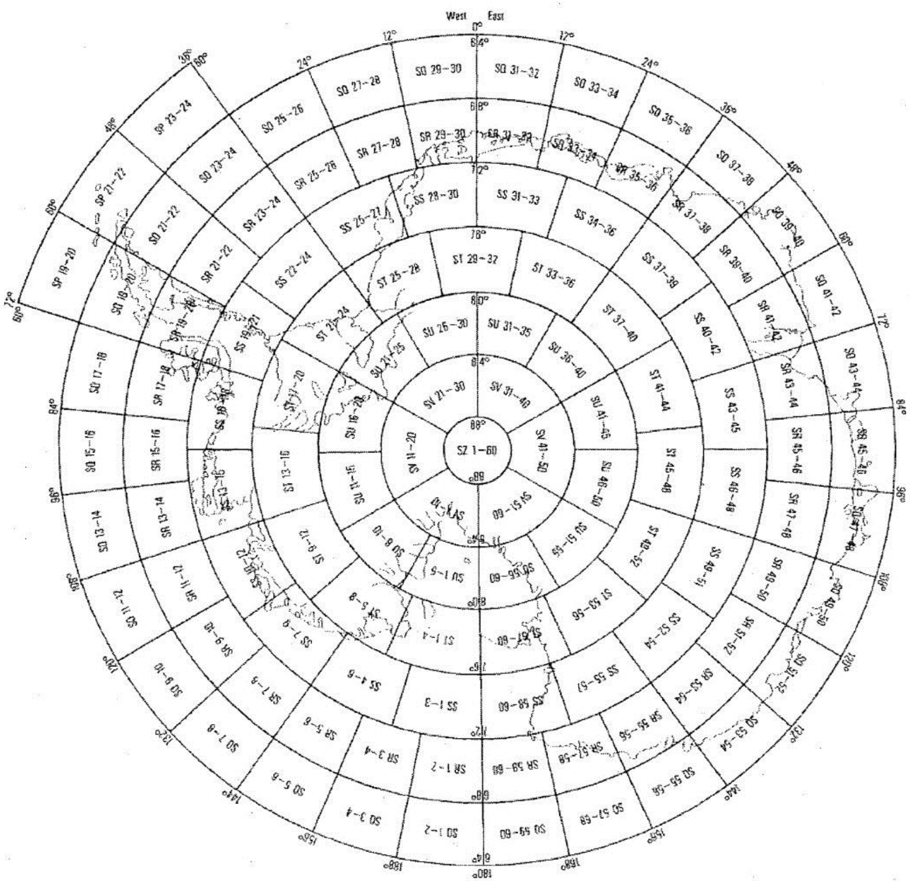
The true scale parallels $\Phi_1$ and $\Phi_2$ of the map sheet representing a territory of $6°\times 4°$, are 40’ to the north of the southern bounding parallel and 40’ to the south of the northern bounding parallel.

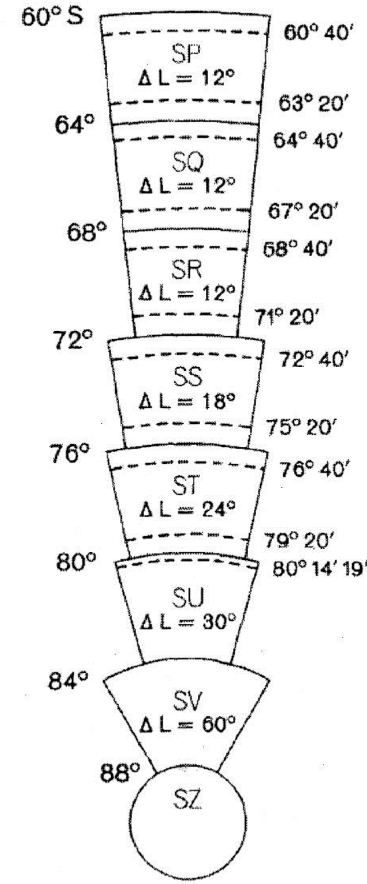
This map system contains maps of larger scale (1 : 500 000), too, whose sheet system flexibly adapts to the location of important air transport centres.

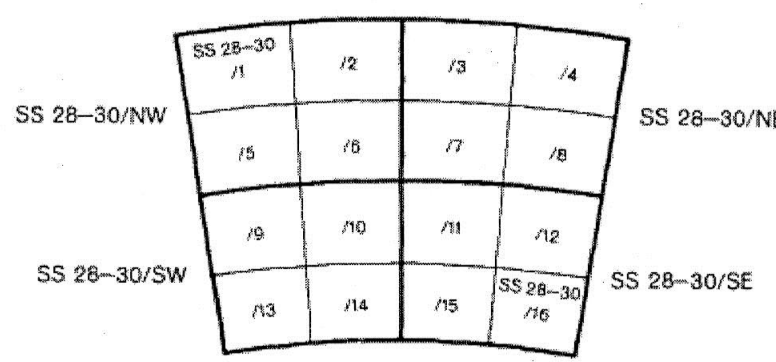
The *topographic mapping* of France, Belgien and countries in northern Africa use the conformal conical projection, too.

The images of *meteorological satellites* are issued in conformal conical.

# Map sheet system of the WAC





# Oblique conformal conical map projection

This version is advantageous for representing areas expanding along a small circle of a sphere. Taking this small circle as the metaparallel of a metacoordinate system with $\varphi_0$, $\lambda_0$ as the metapole, a conformal conical projection should be referred to the metacoordinates $\varphi^*$, $\lambda^*$, where

$$\varphi^* = \arcsin\left(\sin\varphi_0 \cdot \sin\varphi + \cos\varphi_0 \cdot \cos\varphi \cdot \cos(\lambda - \lambda_0)\right)$$

$$\lambda^* = \arcsin\left(\frac{\cos\varphi \cdot \sin(\lambda - \lambda_0)}{\cos\varphi^*}\right)$$

Křovák projection: a double projection composed of a conformal ellipsoid-sphere transformation from the S-JTSK geodetic datum (based on the Bessel ellipsoid) onto the aposphere with the true scale parallel of $\Phi_s$=49°30’, and an oblique conformal conical projection of the aposphere onto the plane. The coordinates of the metapole are: $\varphi_0$= 59°42′ 42.69689″, $\lambda_0$ =42° 31′ 31.41725″ Ferro (~24°50’ Greenwich). The metaparallel with linear scale 0.9999 is given: $\varphi_s$*=78°30’. It was used for the cadastral maps and large scale topographic maps of the former Czechoslovakia from 1927.