LESSON 1

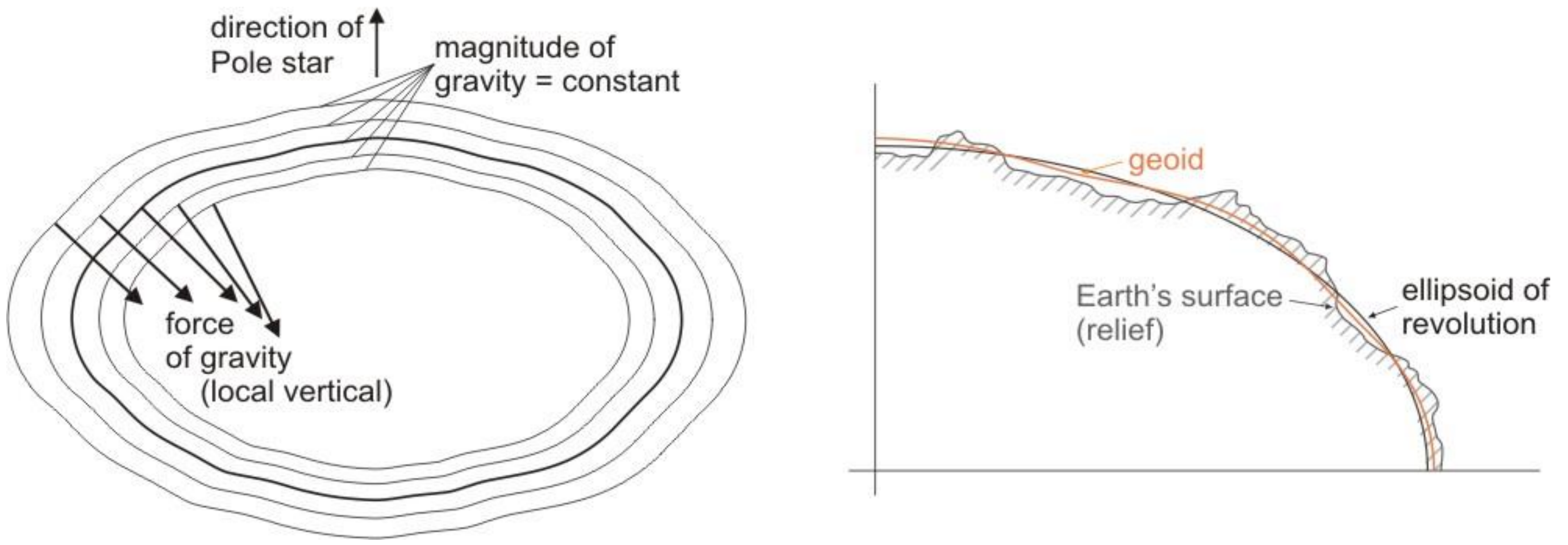
- Process of the mapping from the Earth to the map plane by intermediate surfaces
- The basics of the surface of ellipsoid of revolution
- Geographic coordinate system on spherical surface
- The geometric dimension of graticule lines and notable parts on the sphere of radius R

The approximation of the physical earth's surface by a reference surface satisfying the conditions

The task is to represent the physical Earth surface on a map.

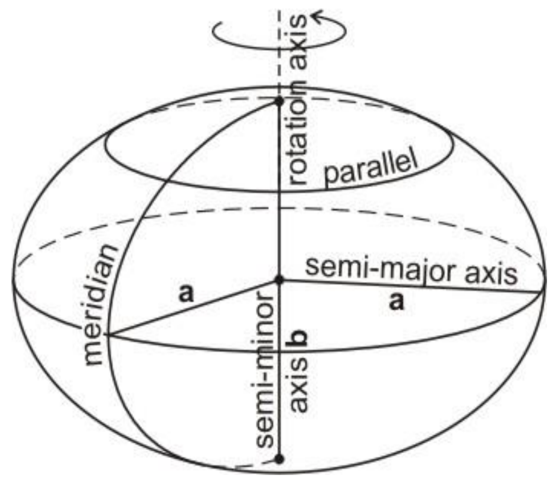
1. The irregular Earth’surface is projected orthogonally onto an intermediate surface (*geoid*) for eliminating the elevation;

   (**Geoid**: the equipotential surface of the force of gravity, coinciding with the sea level assumed to be at rest; it is an irregular surface)



2. The geoid is approached by a continuous and regular surface, which is describable by mathematical formulae.

The basic properties of the surface of ellipsoid of revolution (spheroid)



- a – semi-major axis, radius of the Equator
- b – semi-minor axis
- f – flattening $f = \frac{a-b}{a}$
- e – first eccentricity $e = \sqrt{\frac{a^2 - b^2}{a^2}}$ and $e^2 = \frac{a^2 - b^2}{a^2}$
- e' – second eccentricity $e' = \sqrt{\frac{a^2 - b^2}{b^2}}$

Generally **a** and **1/f** („inverse flattening”) are given, then **e, e^2**, **b** and **e’** can be calculated

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{(a-b)}{a} \cdot \frac{(a+b)}{a} = f \cdot \frac{(2a - a + b)}{a} = .$$

$$= f \cdot \frac{2a - (a-b)}{a} = f \cdot \left(2 - \frac{(a-b)}{a}\right) = f \cdot (2 - f)$$

$$b = a \cdot (1 - f)$$

E.g. **WGS84** ellipsoid (introduced in 1984)

a = 6 378 137.0 m 1/f = 298.257 223 563

f = 1/298.257 223 563

e^2 = ? e = ? b = ? e’ = ?

Some other current ellipsoids:

- Airy 1830 a = 6 377 563.396m; 1/f = 299.324 964 6
- Bessel 1841 a = 6 377 397.155 m; 1/f = 299.152 812 8
- Clarke 1866 a = 6 378 206.4 m; 1/f = 294.9786982
- Clarke 1880 a = 6 378 249.145 m; 1/f = 293.465
- International 1924 (Hayford)

  a = 6 378 388.0 m; 1/f = 297.0
- Krasovskiy a = 6 378 245.0 m; 1/f = 298.3
- IUGG67 a = 6 378 160.0 m; 1/f = 298.247 167 427
- GRS80 a = 6 378 137.0 m; 1/f = 298.257 222 101

Home work: calculate the other characteristics for one of the upper ellipsoids

Geometric surfaces satisfying the conditions for approximation of the geoid

- **Plane** (in the case of a territory <10 km²)

$$A \cdot x + B \cdot y + C \cdot z + D = 0$$

- **Sphere** (in the case of maps of scale < 1 : 1 million)

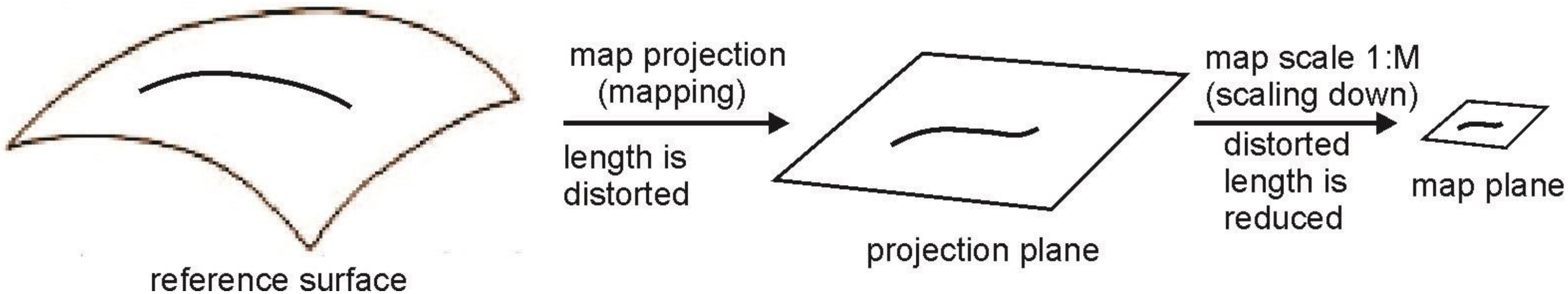
$$x^2 + y^2 + z^2 - R^2 = 0$$ or $$\frac{x^2}{R^2} + \frac{y^2}{R^2} + \frac{z^2}{R^2} - 1 = 0$$

- **Ellipsoid of revolution** (spheroid)

  It arises by the rotation of a half-ellipse about its minor axis

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} - 1 = 0$$

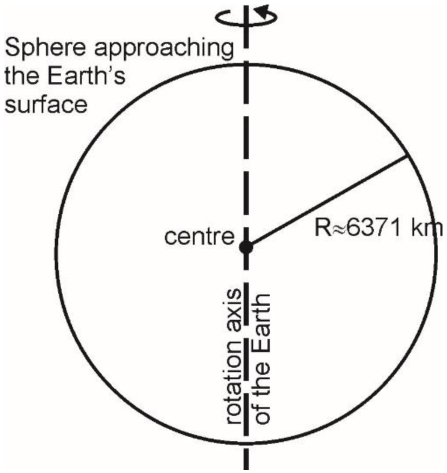
Process of the mapping from the Earth to the map plane by intermediate surfaces

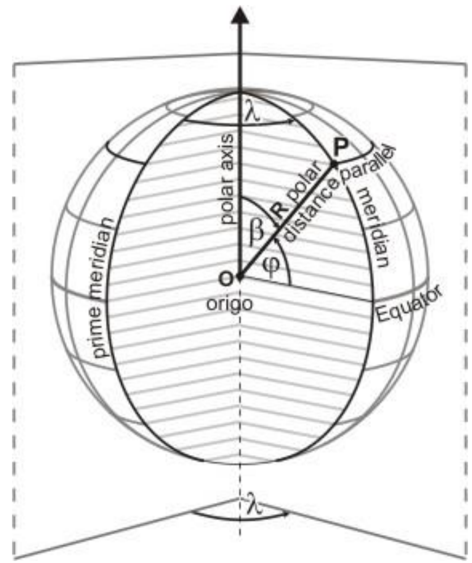


OPERATION	FROM	TO
Orthogonal projection	**Physical Earth surface** (real) (irregular)	**Geoid** (imaginary) (irregular)
fitting an approximate reference surface	**Geoid** (imaginary) (irregular)	**Ellipsoid of revolution** (imaginary) (regular)
mapping by map projection	**Ellipsoid of revolution** (imaginary) (regular)	**Plane of projection** (imaginary) (regular)
scaling down	**Plane of projection** (imaginary) (regular)	**Map plane** (real) (regular)

Geographic coordinate system on spherical surface

- The earth surface can be approximated by a sphere of radius R. A **spatial polar** coordinate system will be established with the origin in the centre of the sphere, and its polar axis coincides with the natural rotation axis of the sphere.





- The polar coordinates of the point P of spherical surface:
  - ρ **polar distance**: is equal to R at every point of the spherical surface, so it can be **omitted**
  - β **polar angle:** its complementary angle: $\varphi=90°-\beta$ is the **geographic latitude**
  - λ **azimuthal angle**: the angle between the initial semi-plane and the semi-plane containing the point P; it is the **geographic longitude**

The graticule of geographic coordinate system on the spherical surface

Coordinate lines: one of the geographic coordinates φ and λ is fixed

φ = constant: **parallel** (circle of latitude) – small circle on sphere
(the latitude is considered as positive on the Northern hemisphere)

λ = constant: **meridian** (circle of longitude) – great circle on sphere
(the longitude is considered as positive on the Eastern hemisphere)

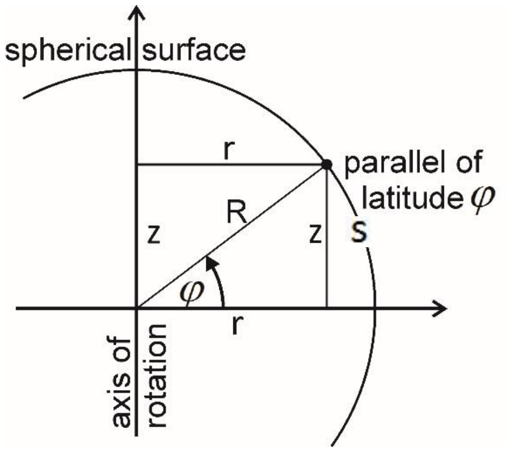
$\varphi = -90°$ South Pole
$\varphi = 0°$ Equator (great circle!)
$\varphi = +90°$ North Pole

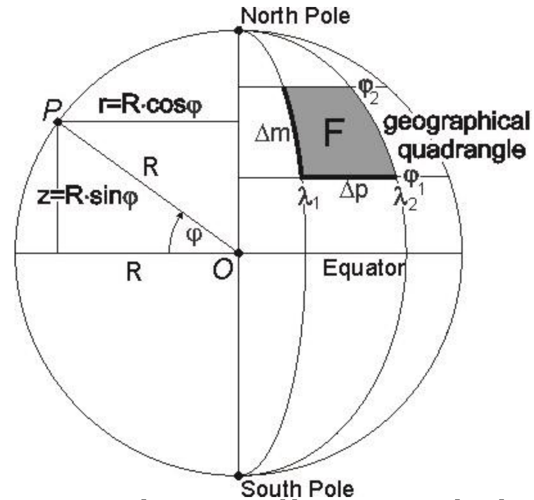
λ = 0° prime meridian
λ = ±180° Date Line
(opposite meridian of the prime meridian)

parallels + meridians = **graticule**

Geometric dimensions of the graticule lines of sphere with radius R (1)

- *r*: radius of the parallel with latitude φ
- *z*: distance of the parallel from plane of the Equator with latitude φ





$$r = R \cdot \cos\varphi$$

$$z = R \cdot \sin\varphi$$

- Notations: arcδ and δ° denote the radian and degree measure of the arbitrary angle δ
- Using these notations: $$arc\delta = \delta^\circ \cdot \frac{\pi}{180^\circ}$$

  Length Δ of the circular arc belonging to the central angle δ: $\Delta = \rho *$arcδ

  where ρ is the radius of the circular arc
- Length of the circular arc s of a meridian between 0° and φ: s = R∗arcφ

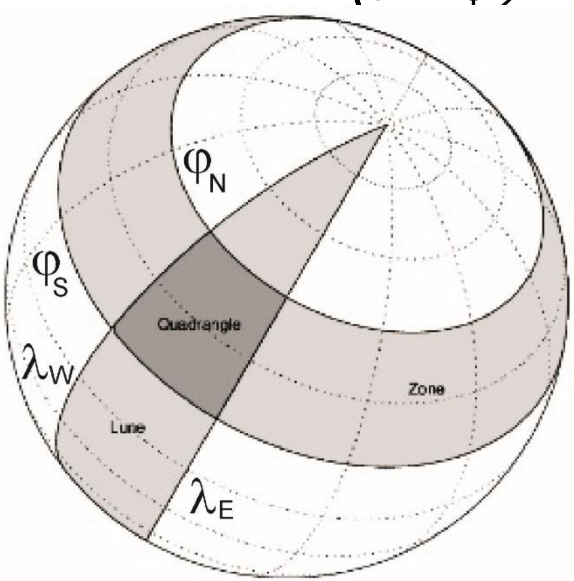
Geometric dimensions of the graticule lines on sphere with radius R (2)

- Applying this formula to the parallel arc Δp between the longitudes λ_1 and λ_2: $\Delta p = r \cdot arc(\lambda_2 - \lambda_1) = R \cdot cos\, \varphi \cdot arc(\lambda_2 - \lambda_1)$

- Simil $\Delta m = R \cdot arc\left(\varphi_2 - \varphi_1\right)$ ıeridian arc Δm:

- The distance Δz between the plane of the parallels φ_1 and φ_2:

$$\Delta z = R\,(\sin \varphi_2 - \sin \varphi_1)$$



Notable parts of the spherical surface:

Spherical zone: a part of the spherical surface bounded by two parallels;

Spherical lune: a part of the spherical surface bounded by two meridians

Geographic quadrangle: bounded by

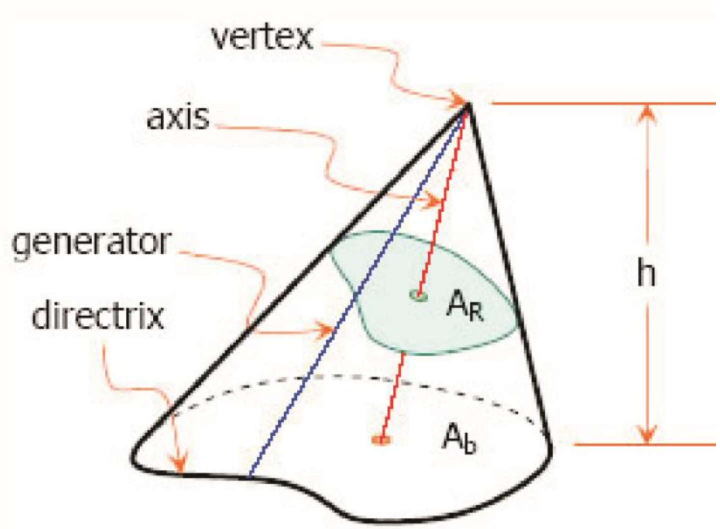
Surface area of the spherical zone, geographic quadrangle and spherical lune (1)

Knowledges to be applied:

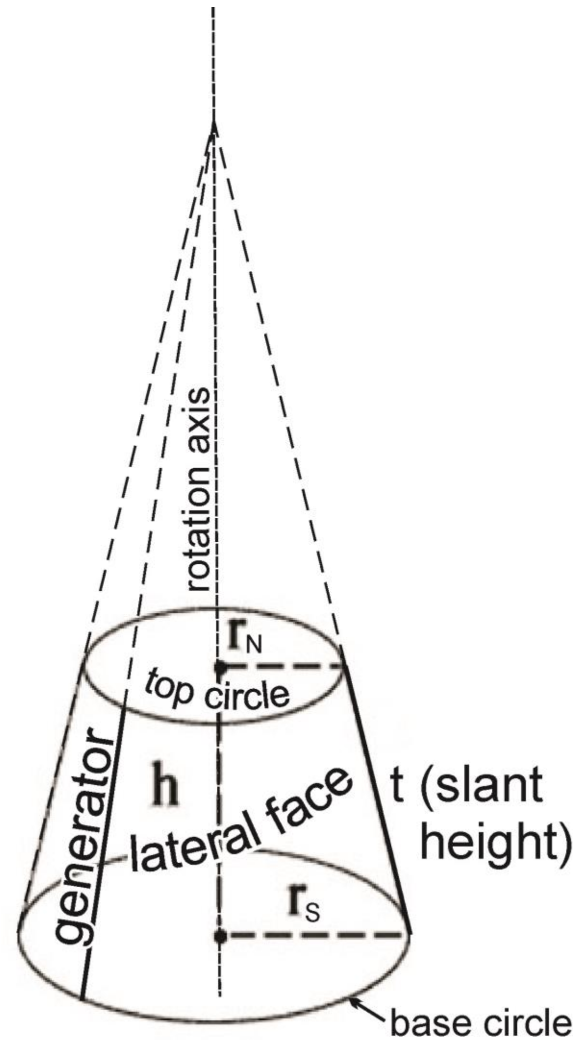
radius ρ, central angle ω

length *s* of circular arc: $s = \rho \cdot arc\omega$

area *F* of circular sector: $F = \frac{\rho^2 \cdot arc\omega}{2}$



cone in general



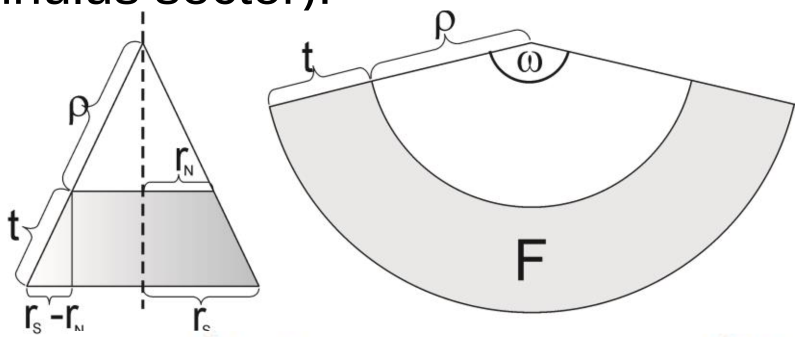
right circular truncated cone

Surface area of the spherical zone, geographic quadrangle and spherical lune (2)

Lateral surface area of a **right circular truncated cone:** cutting the lateral face (nappe) of the truncated cone along a generator (slant height = t) and laying it flat, the area of the ring sector (annulus sector):

- r_S : radius of base circle
- r_N : radius of top circle

t: slant height of completing cone



similar triangles: $\frac{\rho}{r_N} = \frac{t}{r_S - r_N}$ $\quad \rho$ expressed: $\rho = \frac{t \cdot r_N}{r_S - r_N}$, and $\rho + t = \frac{t \cdot r_S}{r_S - r_N}$

ω central angle (in radian): arc $\omega = \frac{2 \cdot r_N \cdot \pi}{\rho} = \frac{2 \cdot \pi}{t} \cdot (r_S - r_N)$

area *F* is the difference of area the circular sectors with radii ρ and ρ+t:

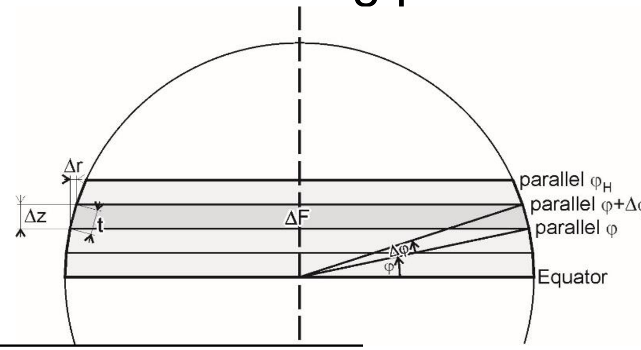
$F_\rho = \frac{t^2 \cdot r_N^2}{(r_S - r_N)^2} \cdot \frac{2 \cdot \pi}{t} \cdot \frac{(r_S - r_N)}{2} = \frac{t \cdot \pi \cdot r_N^2}{r_S - r_N}$ and $F_{\rho+t} = \frac{t^2 \cdot r_S^2}{(r_S - r_N)^2} \cdot \frac{2 \cdot \pi}{t} \cdot \frac{(r_S - r_N)}{2} = \frac{t \cdot \pi \cdot r_S^2}{r_S - r_N}$

area of ring sector: $F = F_{\rho+t} - F_\rho = t \cdot \pi \cdot \frac{r_S^{\ 2} - r_N^{\ 2}}{r_S - r_N} = t \cdot \pi \cdot (r_S + r_N)$

Surface area of the spherical zone bounded by the Equator and the parallel of latitude φ_H

The spherical zone bounded by the Equator and parallel φ_H will be partitioned by parallels into narrow spherical zones, and their surface area will be approached by nappe of truncated cones. The bounding latitudes of such a narrow nappe: φ and $\varphi+\Delta\varphi$; the Δz distance of plane of bounding parallels:

$$\Delta z = R \cdot \left[\sin(\varphi + \Delta\varphi) - \sin\varphi\right]$$



difference Δr of radii of bounding parallels:

$$\Delta r = R \cdot \left[\cos\varphi - \cos(\varphi + \Delta\varphi)\right]$$

slant height t from the pythagorean theorem:

$$t = \sqrt{R^2 \cdot \left[\sin(\varphi + \Delta\varphi) - \sin\varphi\right]^2 + R^2 \cdot \left[\cos(\varphi + \Delta\varphi) - \cos\varphi\right]^2}$$

approaching the surface area of the narrow spherical zone by the nappe of truncated cone:

$$\Delta F \approx R \cdot \sqrt{\left[\sin(\varphi + \Delta\varphi) - \sin\varphi\right]^2 + \left[\cos(\varphi + \Delta\varphi) - \cos\varphi\right]^2} \cdot \pi \cdot R \cdot \left[\cos\varphi + \cos(\varphi + \Delta\varphi)\right] =$$

$$= R^2 \cdot \pi \cdot \sqrt{\left[\frac{\sin(\varphi + \Delta\varphi) - \sin\varphi}{\Delta\varphi}\right]^2 + \left[\frac{\cos(\varphi + \Delta\varphi) - \cos\varphi}{\Delta\varphi}\right]^2} \cdot \left[\cos\varphi + \cos(\varphi + \Delta\varphi)\right] \cdot \Delta\varphi$$

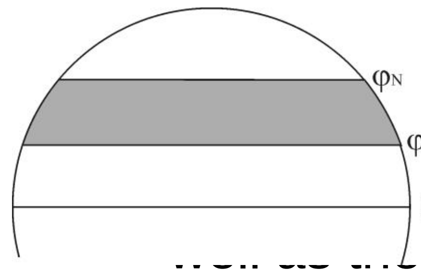
Surface area F_H of a spherical zone, bounded by the Equator and the parallel of latitude j_H, as well as the geographic quadrangle and spherical lune

The surface area F_H of **spherical zone**, bounded by the **Equator** and the **parallel of latitude φ_H**, can be approached by the sum of surface area of narrow nappes. During the refinement of partition, this sum converges to the integral:

$$F_H \approx \sum \Delta F \approx \sum R^2 \cdot \pi \cdot \sqrt{\cos^2 \varphi + \sin^2 \varphi} \cdot 2 \cdot \cos\varphi \cdot \Delta\varphi \rightarrow 2 \cdot R^2 \cdot \pi \cdot \int_0^{\varphi_H} \cos\varphi \, d\varphi = 2 \cdot R^2 \cdot \pi \cdot \sin\varphi_H$$

The surface area F_z of the **spherical zone**, bounded by the **parallels φ_S and φ_N**:

$$F_z = 2 \cdot R^2 \cdot \pi \cdot \left(\sin\varphi_N - \sin\varphi_S\right)$$



The surface area $F_q(\varphi_S, \varphi_N, \lambda_W, \lambda_E)$ of the **geographic** **ngle** bounded by the parallels φ_S and φ_N, as meridians λ_W and λ_E, is proportional part of the surface area F_z:

$$F_q(\varphi_S, \varphi_N, \lambda_W, \lambda_E) = R^2 * (\sin\varphi_N - \sin\varphi_S) * \mathrm{arc}(\lambda_E - \lambda_W)$$

The surface area F_L of **spherical lune** between the longitudes λ_W and λ_E is the proportional part of the surface area $4 * R^2 * \pi$ of the sphere:

$$F_L = 2 * R^2 * \mathrm{arc}(\lambda_E - \lambda_W)$$

Examples:

- Budapest (Control point 1) coordinates:

  latitude φ = 47°28'29.262" (radian=?)

  longitude (prime meridian: Greenwich) λ = 19°3'43.303" (radian=?)
- Length Δm of the meridian arc between the Equator and the Budapest point on an approximate sphere (R = 6371000m) =?
- Length Δp of the parallel arc between the Bp point and the prime meridian =?
- Distance Δz between the parallel crossing Budapest point and the Equator =?
- Extra credit example: the (perpendicular) distance of the Budapest point from the prime semi-plane (Greenwich) =?
- Surface area of the spherical zone between 40° and 50° North =?
- Surface area of the spherical lune between 10° and 20° East =?
- Surface area of the geographic quadrangle bounded by the Equator and Budapest point parallel, as well as the prime meridian and Bp point meridian =?
- **Home work**: the same calculations for a GPS point in your city