Lesson 2

- Graticule lines on the ellipsoid of revolution and their geometric dimensions

- Notable radii of curvature of the surface lines on reference surfaces

Geographic coordinate system on ellipsoidal surface

Latitudes on the ellipsoid of revolution

1. Spatial polar coordinate system with its origin in the centre of ellipsoid, and its polar axis coincides with the ellipsoid's rotation axis corresponding to the Earth's rotation axis

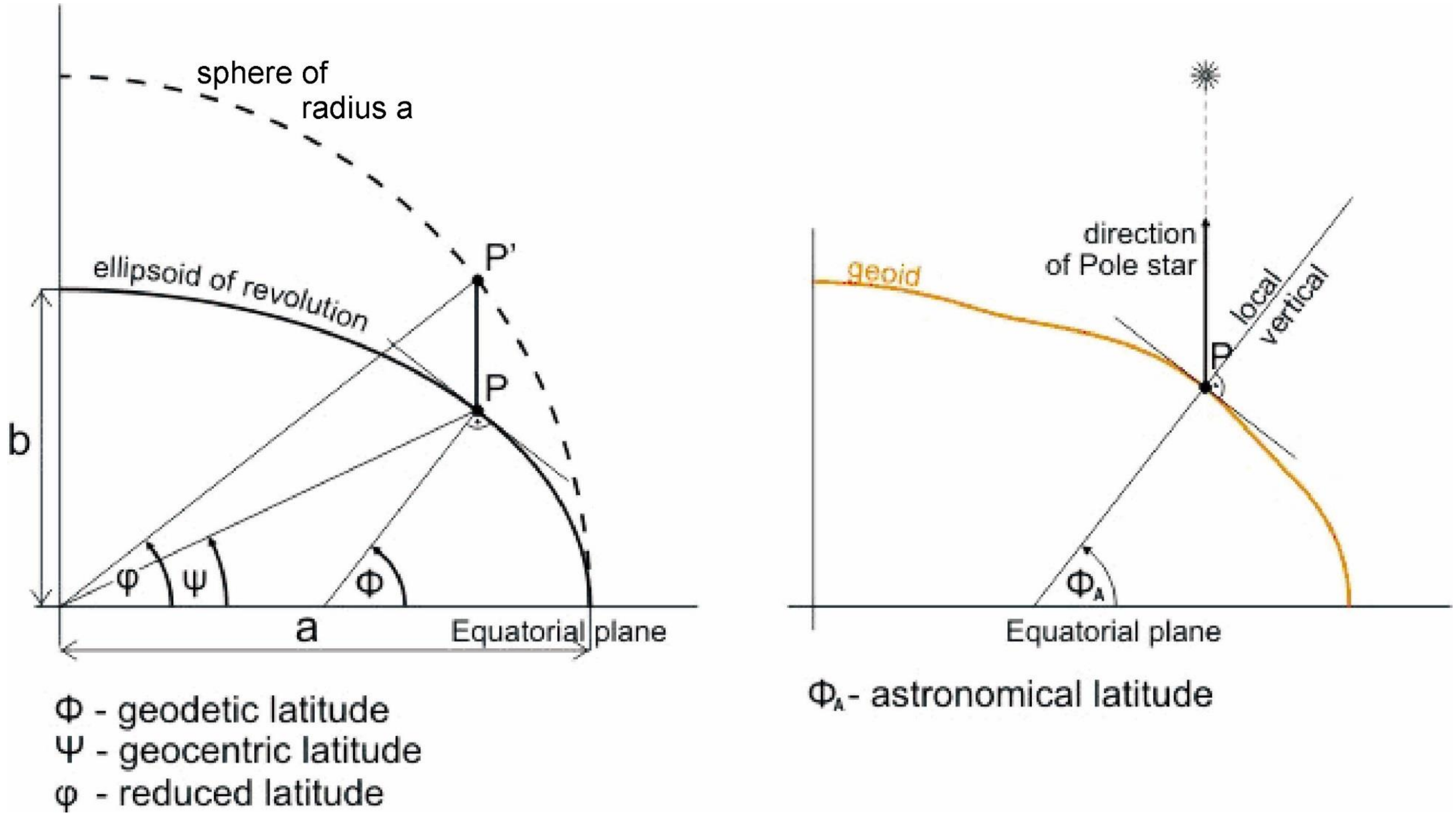
   Ψ $(= 90° - \mathrm{B})$ „*geocentric latitude*" (B is the polar angle)

2. Let the ellipsoid be transformed into a sphere by an affinity perpendicularly to the plane of the Equator, with a ratio of *a/b* . Taking the usual spatial polar coordinates on this sphere of radius *a*, then φ $(= 90° - \beta)$ is the „*reduced latitude*" (β is the polar angle)

3. Φ **„*geodetic latitude*" is the angle between the equatorial plane and the direction perpendicular to the ellipsoidal surface (that is perpendicular to the tangent plane of the surface) at the point P.**

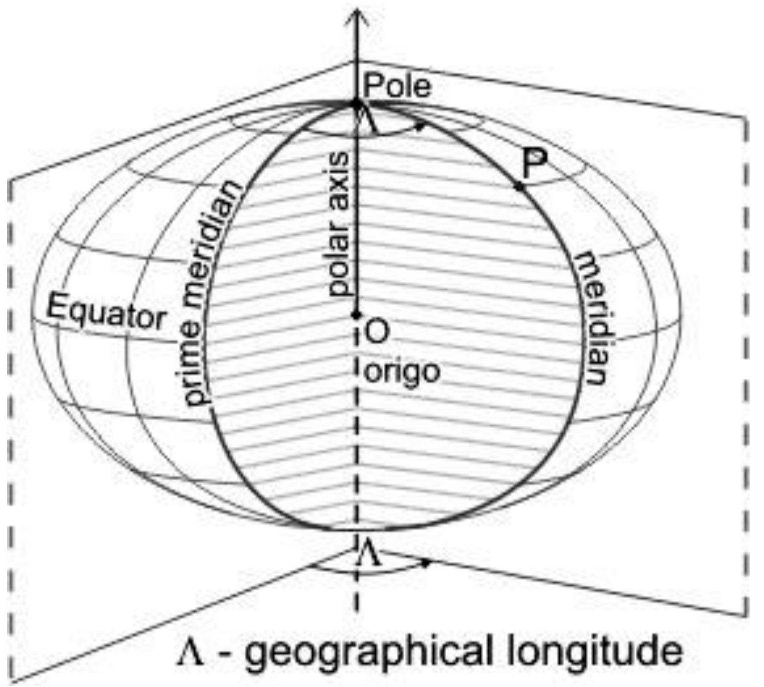
   It can be measured approximately with the help of the angle between the local perpendicular of the geoid (plumb) and the plane, perpendicular to the direction towards the Pole-star („*astronomical latitude*").

Latitudes on the ellipsoid and the geoid



Longitude on the ellipsoid

- **Longitude** Λ on the ellipsoid can be interpreted on the same way, as λ on the sphere – by the **azimuthal polar angle**



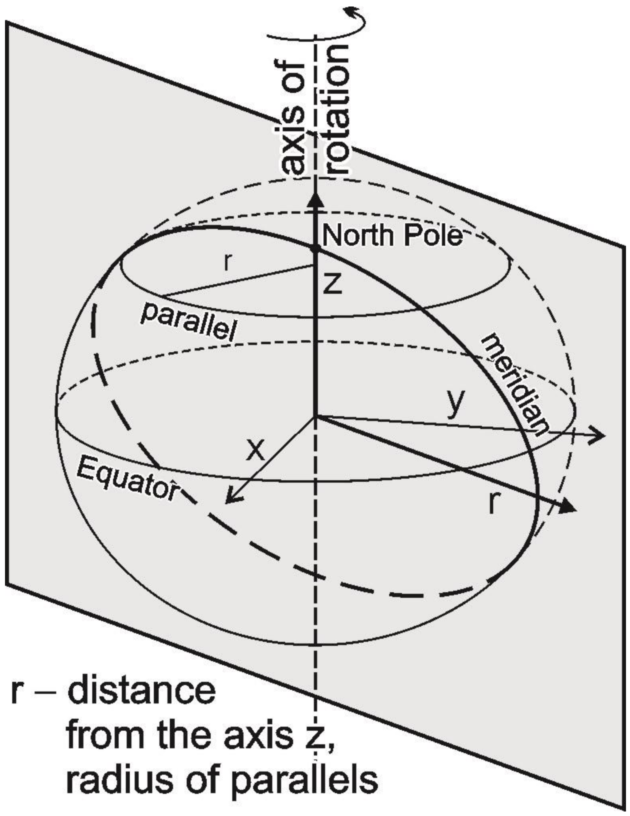
- The longitude difference on the physical Eart's surface can be measured by astronomical tools, with the help of the **time difference of the culmination** on the whereabouts
- 1 hour **time** difference corresponds to 15° **longitude** difference

The graticule on the ellipsoidal surface composed of coordinate lines of the geographic coordinates

- The geodetic latitude Φ and the longitude Λ are considered as the **geographic coordinates** on the ellipsoid. The value of the latitude: $-90° \leq \Phi \leq +90°$, its sign is positive on the northern hemisphere. The value of the longitude: $-180° \leq \Lambda \leq +180°$, its sign is positive on the eastern hemisphere.
- Φ is fixed: **parallel** (circle of latitude); $\Phi = \pm 90°$ *South and North Pole;* $\Phi = 0°$ *Equator,* $\Phi = \pm$ *66°33'49.22" Arctic and Antarctic Circles,* $\Phi = \pm$ *23°26'10.78" Tropic of Cancer, Tropic of Capricorn* (2022).
- Λ is fixed: **meridian** (line of longitude); its sign is positive on the eastern hemisphere; $\Lambda = 0°$ *prime meridian*; $\Lambda = \pm 180°$ *Date Line*.
- *Antimeridian*: a meridian which is opposite to a given meridian (the difference of their longitude is 180°).
- An arbitrary meridian and its antimeridian with a longitude difference of 180° create a *bimeridian* both on the sphere and the ellipsoid, which is a planar curve.
- The prime meridian on the ellipsoidal surface can be local or global, too (Greenwich, Gellérthegy, etc.).

Geometric dimensions of graticule lines on the ellipsoid with semi-axes *a* and *b* (1)

- Let a spatial rectangular coordinate system x, y, z and a plane containing a meridian and with it the axis z be considered. Then a planar rectangular coordinate system r, z can be established in this plane.

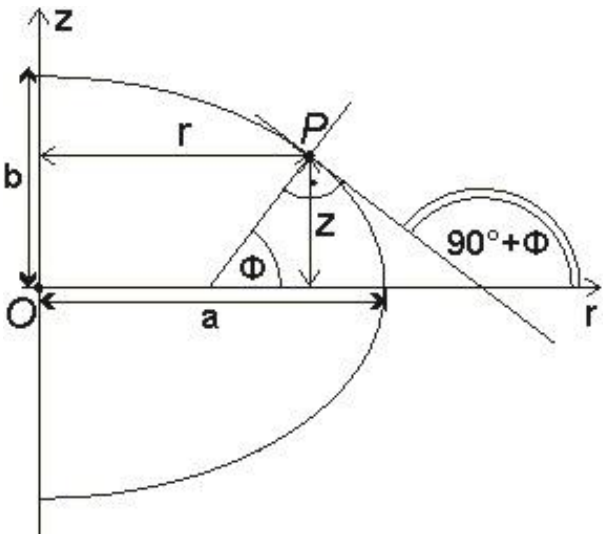


Geometric dimensions of graticule lines on the ellipsoid with semi-axes *a* and *b* (2)

The parallels of the ellipsoid

Completing the meridan into a bimeridian, the equation of this curve with its centre in the origin is, refering to the *r,z* planar coordinate system:

$$\frac{r^2}{a^2}+\frac{z^2}{b^2}-1=0$$



Let z be expressed as function of *r* whose graph coincides the upper part of the ellipse:

$$z=b\cdot\sqrt{1-\frac{r^2}{a^2}}=\frac{b}{a}\cdot\sqrt{a^2-r^2}$$

and its derivative is

$$\frac{dz}{dr}=-\frac{b\cdot r}{a\cdot\sqrt{a^2-r^2}}$$

On the other hand, the derivative means the *slope* of the tangent line, whose angle with the axis *r* is equal to (90°+Φ) at a point P on the parallel Φ. The slope is the *tangent of the angle* (90°+Φ).

That is,

$$\frac{dz}{dr}=\tan\left(90°+\Phi\right)=-\frac{\cos\Phi}{\sin\Phi}$$

Geometric dimensions of graticule lines on the ellipsoid with semi-axes *a* and *b* (3)

- The consequence of the two equalities concerning $\frac{dz}{dr}$:

$$-\frac{b \cdot r}{a \cdot \sqrt{a^2 - r^2}} = -\frac{\cos\emptyset}{\sin\emptyset}$$

The coordinate *r* can be expressed after squaring and rearranging this equation which is the ***radius of the parallel*** of latitude Φ:

$$r = \frac{a^2 \cdot \cos\Phi}{\sqrt{a^2 \cdot \cos^2\Phi + b^2 \cdot \sin^2\Phi}} = \frac{a \cdot \cos\Phi}{\sqrt{1 - e^2 \cdot \sin^2\Phi}}$$

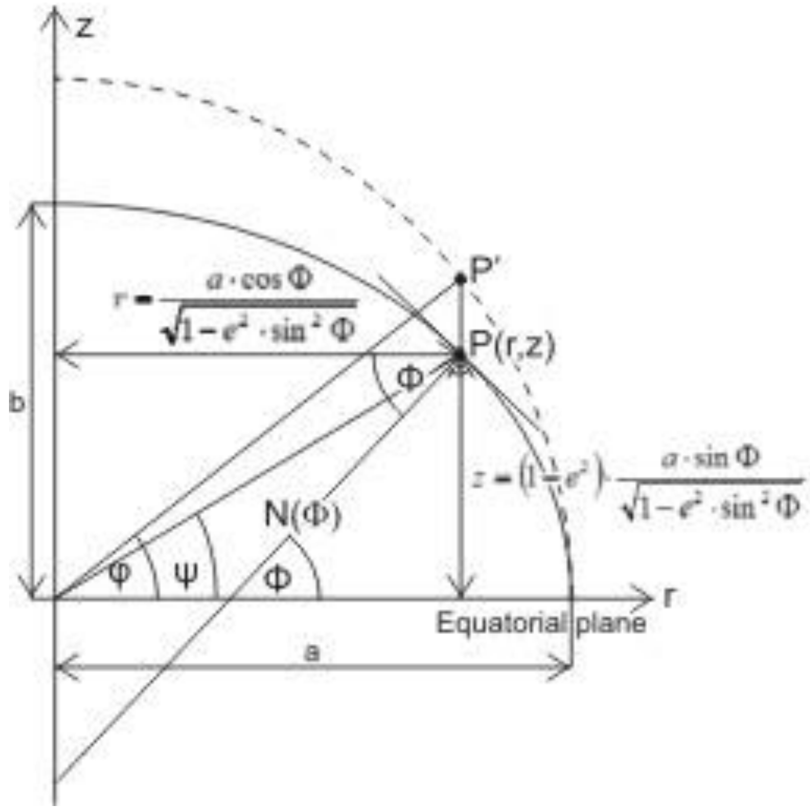
Substituting *r* into the expression of the function, z can be expressed:

$$z = \frac{b}{a} \cdot \sqrt{a^2 - r^2} = \frac{b}{a} \cdot \sqrt{a^2 - \frac{a^4 \cdot \cos^2\emptyset}{a^2 \cdot \cos^2\emptyset + b^2 \cdot \sin^2\emptyset}} = \frac{b}{a} \cdot \sqrt{\frac{a^2 \cdot b^2 \cdot \sin^2\emptyset}{a^2 \cdot \cos^2\emptyset + b^2 \cdot \sin^2\emptyset}}$$

Thus, the ***distance*** *z* between the Equator and the parallel Φ:

$$z = \frac{b^2 \cdot \sin\Phi}{\sqrt{a^2 \cdot \cos^2\Phi + b^2 \cdot \sin^2\Phi}} = \frac{b^2}{a^2} \cdot \frac{a \cdot \sin\Phi}{\sqrt{1 - e^2 \cdot \sin^2\Phi}} = \left(1 - e^2\right) \cdot \frac{a \cdot \sin\Phi}{\sqrt{1 - e^2 \cdot \sin^2\Phi}}$$

Connections between the different latitudes on the ellipsoid



On the basis of the figure:

$$\tan\Psi = \frac{z}{r} = \frac{b^2 \cdot \sin\Phi}{a^2 \cdot \cos\Phi} = \frac{b^2}{a^2} \cdot \tan\Phi$$

Transforming the ellipsoid by affinity with the ratio of *a/b* into a sphere of radius *a*, the position vector of point P' – the arm of the geocentric latitude – passes into the arm of the reduced latitude, that is

$$\tan\varphi = \frac{\frac{a}{b} \cdot z}{r} = \frac{a}{b} \cdot \tan\Psi$$

It follows from these:

$$\tan\varphi = \frac{a}{b} \cdot \tan\Psi = \frac{a}{b} \cdot \frac{b^2}{a^2} \cdot \tan\Phi = \frac{b}{a} \cdot \tan\Phi$$

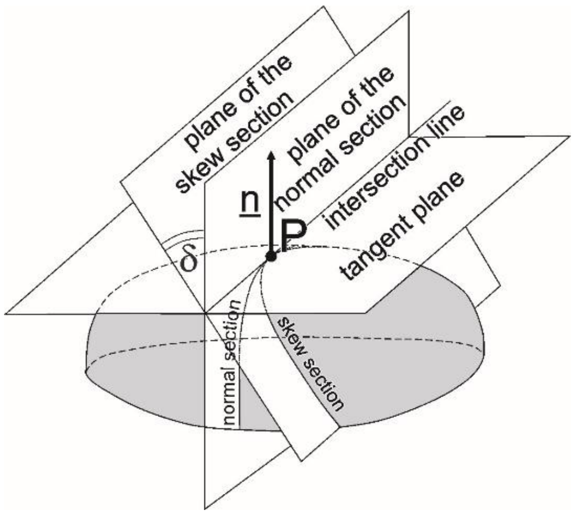
As $\frac{a}{b} > 1$ and $\frac{b}{a} < 1$, the relation among the three latitudes:

$$|\Psi| \le |\varphi| \le |\Phi|$$

N: the radius of curvature normal to the meridian (1)

Let a smooth planar curve S be crossing the point P.

Osculating circle: a circle passing through P, and its first and second derivative are equal to the ones of the curve S at the point P. Its radius is called ***radius of curvature***. (The osculating circle of a circle is itself.)



The ellipsoidal surface is sectioned by planes passing the surface point P.

Normal section: contains the normal vector **n**

Skew section: can be got by rotating the normal section about the intersection line of the tangent plane and the plane of normal section

δ is the angle between the planes of the normal and the skew sections

Let the osculating circles of these sections at P be taken.

Meusnier's theorem: the ratio between the radius of curvature r_s of the skew section and the radius of curvature r_n of the normal secion is equal to the cosine of the angle δ:

$$\cos \delta = \frac{r_s}{r_n}$$

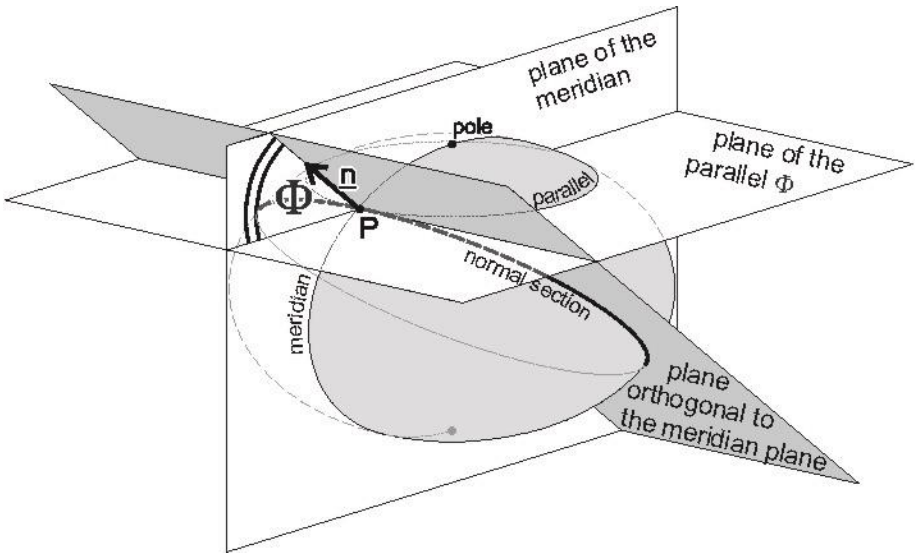
N: the radius of curvature normal to the meridian (2)

Applying the Meusnier's theorem to the ellipsoidal surface:

The plane of the normal section crossing the point P is appointed so that it is perpendicular to the meridian plane containig P

The skew section is created by the plane of the parallel crossing P

The angle between the two planes equals Φ, thus $\cos\Phi = \frac{r(\Phi)}{N}$



Consequently,

$$N(\Phi) = \frac{r(\Phi)}{\cos\Phi} = \frac{a}{\sqrt{1-e^2 \cdot \sin^2\Phi}}$$

N is called *radius of curvature normal to the meridian* (or *prime vertical radius of curvature*).

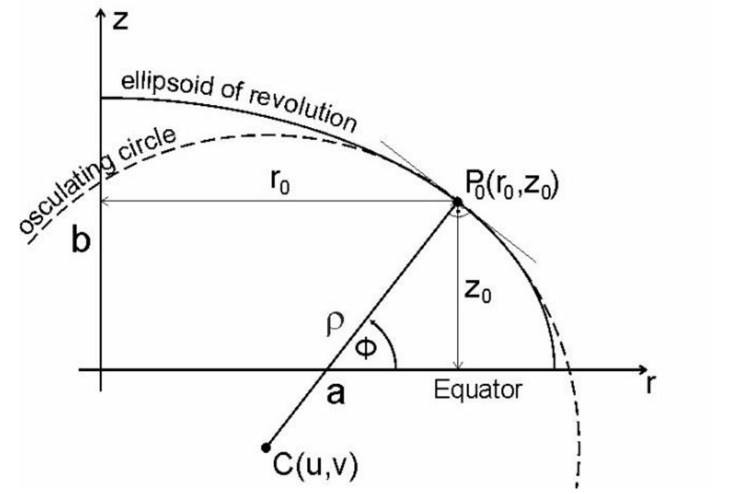
Note: N is the hypotenuse of the right triangle with the leg *r* and angle Φ.

Substituting N into the formulae of *r* and *z*:

$r(\Phi) = N \cdot \cos\Phi$ and $z = \frac{b^2}{a^2} \cdot N(\Phi) \cdot \sin\Phi = (1-e^2) \cdot N(\Phi) \cdot \sin\Phi$

The meridians of the ellipsoid

The meridian is a half-ellipse with semi--axes *a* and *b*. The radius ρ of osculating circle belonging to the point P_0 with latitude Φ_0 is needed.



The formula for the meridian arc as function of *r*:

$$z = \frac{b}{a} \cdot \sqrt{a^2 - r^2}$$

The formula of the osculating circle with radius ρ and with coordinates of the centre point C(u,v):

$$z = \sqrt{\rho^2 - (r-u)^2} + v \qquad \left(\quad x^2 + y^2 - R^2 = 0 \qquad (x-u)^2 + (y-v)^2 - R^2 = 0 \quad \right)$$

The definition of the osculating circle provides three equations for $P_0\,(r_0, z_0)$**:**

$$\sqrt{\rho^2 - (r_0 - u)^2} + v = \frac{b}{a} \cdot \sqrt{a^2 - r_0^{\,2}}$$ $P_0\ (r_0\ , z_0\ )$ is the common point

$$-\frac{r_0 - u}{\sqrt{\rho^2 - (r_0 - u)^2}} = -\frac{b \cdot r_0}{a \cdot \sqrt{a^2 - r_0^{\,2}}}$$ common tangent at $P_0\ (r_0\ , z_0\ )$

$$-\frac{\rho^2}{\left[\rho^2 - (r_0 - u)^2\right]^{\frac{3}{2}}} = -\frac{b \cdot a}{\left[a^2 - r_0^{\,2}\right]^{\frac{3}{2}}}$$ second derivatives equal at $P_0\ (r_0\ , z_0\ )$

The meridian radius of curvature

The solution of this system of equations results in the radius ρ of the osculating circle at the point P of latitude Φ_0 :

$$\rho = \frac{a \cdot \left(1 - e^2\right)}{\left[1 - e^2 \cdot \sin^2 \Phi_0\right]^{\frac{3}{2}}}$$

Its name is ***meridian radius of curvature***, denoted by M and depends on the latitude Φ.

$$M\left(\Phi\right) = \frac{a \cdot \left(1 - e^2\right)}{\left[1 - e^2 \cdot \sin^2 \Phi\right]^{\frac{3}{2}}}$$

Due to the flattening of the ellipsoid, this radius is shortest on the Equator, and longest at the Pole.

Example: the meridian radius of curvature M and the radius of curvature normal to the meridian N values on the WDS84 ellipsoid ($\Phi = 0°$ and $\Phi = 90°$)

Dimensions of the graticule on the ellipsoidal surface

Length Δp of parallel arc on latitude Φ between Λ_1 and Λ_2 :

$$\Delta p = \frac{a \cdot \cos\Phi}{\sqrt{1-e^2 \cdot \sin^2\Phi}} \cdot arc(\Lambda_2 - \Lambda_1)$$

Length Δm of meridian arc between Φ_1 and Φ_2 :

- Let the interval (Φ_1 , Φ_2) be equipartited into n subintervals, and their length $\Delta\Phi_i$ multiplied by M(Φ_i) on intermediate latitudes Φ_i .
- After summarizing the products, make the partition of the interval denser and denser, than the sum converges to the integral:

$$\Delta m \approx \sum_i M(\Phi_i) \cdot \Delta\Phi_i \rightarrow \int_{\Phi_1}^{\Phi_2} M(\Phi)\, d\Phi \quad \text{that is} \quad \Delta m = \int_{\Phi_1}^{\Phi_2} \frac{a \cdot (1-e^2)}{\left[1-e^2 \cdot \sin^2\Phi\right]^{\frac{3}{2}}} d\Phi$$

(calculated numerically, by trapezoidal or Simpson's rule)

The distance Δz between the parallels of Φ_1 and Φ_2 :

$$\Delta z = (1-e^2) \cdot [N(\emptyset_2) \cdot sin\emptyset_2 - N(\emptyset_1) \cdot sin\emptyset_1]$$

Examples:

- Budapest (Control point 1) WGS84 coordinates:

  latitude Φ = 47°28'29.262" (radian=?)

  longitude Λ = 19°3'43.303" (radian=?)
- Length Δm of the meridian arc between the Equator and the Budapest point on the WGS84 ellisoid =?
- Length Δp of the parallel arc between the Bp point and the Greenwich prime meridian =?
- Distance Δz between the parallel crossing Bp point and the Equator =?
- Extra credit example: the (perpendicular) distance of the Bp point from the Greenwich prime semi-plane =?

- **Home work**: the same calculations for a GPS point in your city