Lesson 3

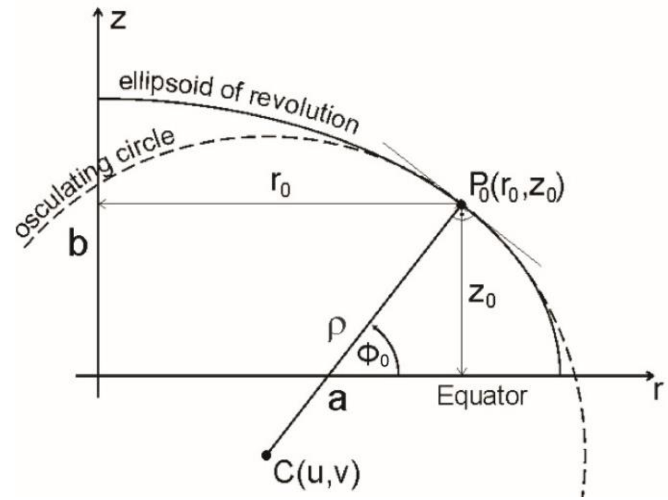
- Meridian on the ellipsoid, meridian radius of curvature M

- Geometric dimensions of the graticule lines and the notable parts of ellipsoidal surface

- Transformations between the spatial rectangular and geographic coordinates

The meridians of the ellipsoid

The meridian is a half-ellipse with semi--axes *a* and *b*. The radius ρ of osculating circle of this ellipse, belonging to the point P_0 with latitude Φ_0 is needed.



The formula for the meridian arc as function of *r* :

$$z = \frac{b}{a} \cdot \sqrt{a^2 - r^2}$$

The formula of the osculating circle with radius ρ and with coordinates of the centre point C(u,v): ($x^2 + y^2 - R^2 = 0 \qquad (x-u)^2 + (y-v)^2 - R^2 = 0$)

$$z = \sqrt{\rho^2 - (r-u)^2} + v$$

The definition of the osculating circle provides three equations for P_0 (r_0, z_0)**:**

$$\sqrt{\rho^2 - (r_0 - u)^2} + v = \frac{b}{a} \cdot \sqrt{a^2 - r_0^{\,2}}$$ P_0 $(r_0 \; , z_0 \;)$ is the common point

$$-\frac{r_0 - u}{\sqrt{\rho^2 - (r_0 - u)^2}} = -\frac{b \cdot r_0}{a \cdot \sqrt{a^2 - r_0^{\,2}}}$$ common tangent at P_0 $(r_0 \; , z_0 \;)$

$$-\frac{\rho^2}{\left[\rho^2 - (r_0 - u)^2\right]^{\frac{3}{2}}} = -\frac{b \cdot a}{\left[a^2 - r_0^{\,2}\right]^{\frac{3}{2}}}$$ second derivatives equal at P_0 $(r_0 \; , z_0 \;)$

The meridian radius of curvature

The solution of this system of equations for the radius ρ of the osculating circle with radius r_0 of the parallel crossing the point P_0 **:**

$$\rho = \frac{[a^4 - {r_0}^2 \cdot (a^2 - b^2)]^{\frac{3}{2}}}{a^4 \cdot b}$$

Substitute the expression

$$r_0 = \frac{a \cdot cos\emptyset_0}{\sqrt{1 - e^2 \cdot cos^2\emptyset_0}}$$

into the formula for ρ above, which results in the ***meridian radius of curvature*** on the latitude Φ denoted by M:

$$M(\Phi) = \frac{a \cdot (1 - e^2)}{\left[1 - e^2 \cdot \sin^2 \Phi\right]^{\frac{3}{2}}}$$

Due to the flattening of the ellipsoid, this radius is shortest on the Equator, and longest at the Pole.

Example: calculate the meridian radius of curvature M values ($\Phi = 0°$, Bp point latitude and $\Phi = 90°$) and compare them to the radius of curvature N normal to the meridian values on the WDS84 ellipsoid (a = 6378137 m, 1/f=298.257223563).

Dimensions of the graticule on the ellipsoidal surface

Length Δp of parallel arc on latitude Φ between the longitudes Λ_1 and Λ_2 :

$$\Delta p = \frac{a \cdot \cos\Phi}{\sqrt{1-e^2 \cdot \sin^2\Phi}} \cdot arc(\Lambda_2 - \Lambda_1)$$

The distance Δz between the parallels of Φ_1 and Φ_2 :

$$\Delta z = (1-e^2) \cdot [N(\emptyset_2) \cdot sin\emptyset_2 - N(\emptyset_1) \cdot sin\emptyset_1]$$

Length Δm of meridian arc between the latitudes Φ_1 and Φ_2 :

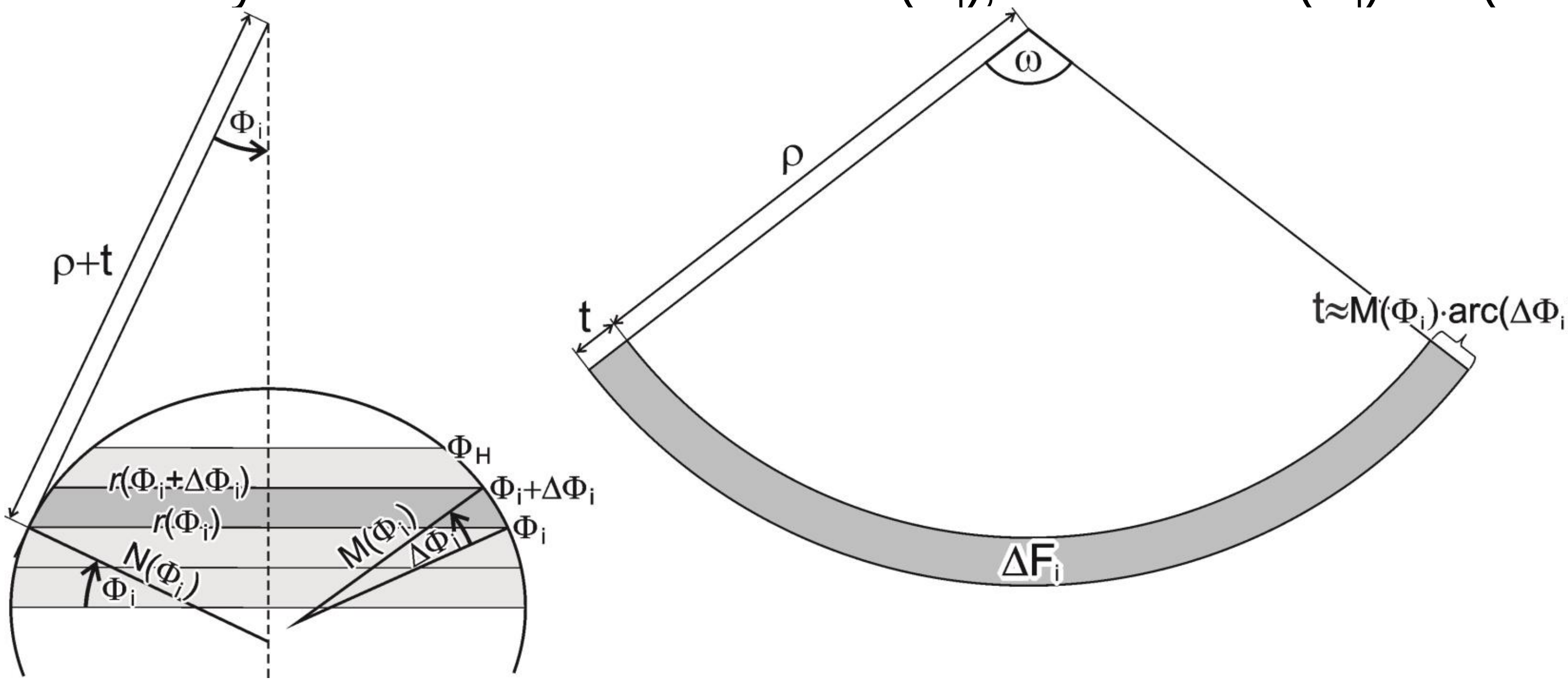
- Let the interval (Φ_1, Φ_2) be equipartited into n subintervals, and their length $\Delta\Phi_i$ multiplied by $M(\Phi_i)$ on intermediate latitudes Φ_i .
- After summarizing the products, make the partition of the interval denser and denser, than the sum converges to the integral:

$$\Delta m \approx \sum_i M(\Phi_i) \cdot \Delta\Phi_i \to \int_{\Phi_1}^{\Phi_2} M(\Phi)\, d\Phi \quad \text{that is} \quad \Delta m = \int_{\Phi_1}^{\Phi_2} \frac{a \cdot (1-e^2)}{\left[1-e^2 \cdot \sin^2\Phi\right]^{\frac{3}{2}}} d\Phi$$

(calculated numerically, by trapezoidal or Simpson’s rule)

Surface area of objects on the ellipsoidal surface (1)

Similarly to the spherical zone (Lecture_2), the **ellipsoidal zone** bounded by the Equator and parallel Φ_H will be partitioned into narrow *ellipsoidal zones*, and their surface area will be approached by the *lateral face* of a truncated cone, with the same base and top circles. The bounding parallels of such a narrow truncated cone are Φ_i and $\Phi_i+\Delta\Phi_i$, their radius $r(\Phi_\iota)$ and $r(\Phi_\iota+\Delta\Phi_\iota)$. The *slant height* t is near the length of a meridian arc between Φ_i and $\Phi_i+\Delta\Phi_i$ approached by a circular arc with radius $M(\Phi_i)$, that is $t \approx M(\Phi_i) * arc(\Delta\Phi_i)$.



Hence, $\Delta F_i = t*\pi*[r(\Phi_\iota)+r(\Phi_\iota+\Delta\Phi_\iota)] = M(\Phi_i)*\Delta\Phi_i*\pi*[N(\Phi_i)*\cos\Phi_i+N(\Phi_i+\Delta\Phi_i)*\cos(\Phi_i+\Delta\Phi_i)]$

Summarizing: $F_{sum} = \Sigma\, \Delta F_i = \Sigma\, 2*\pi*M*N(\Phi_i)*\cos\Phi_i*\Delta\Phi_i = \sum_i \frac{2 \cdot \pi \cdot a^2 \cdot (1-e^2) \cdot \cos\emptyset}{(1-e^2 \cdot \sin^2\emptyset)^2} \cdot \Delta\emptyset_i$

Surface area of objects on the ellipsoidal surface (2)

- During the refinement of resolution, F_{sum} converges to the integral:

$$F_{sum} \rightarrow a^2 \cdot (1 - e^2) \cdot \pi \cdot \int_0^{\emptyset_H} \frac{2 \cdot cos\emptyset}{(1 - e^2 \cdot sin^2\emptyset)^2} \, d\emptyset$$

- After carrying out the integration, the surface area F_Φ of an *ellipsoidal zone* between the Equator and the parallel of latitude Φ is:

$$F_\Phi = a^2(1-e^2)\pi\left(\frac{\sin\Phi}{1-e^2\cdot\sin^2\Phi} + \frac{1}{2e}\cdot\ln\frac{1+e\cdot\sin\Phi}{1-e\cdot\sin\Phi}\right)$$

- Consequently, the surface area F of the *geographic quadrangle* buonded by the parallels Φ_1 and Φ_2, as well as the meridians Λ_1 and Λ_2 :

$$F = a^2(1-e^2)\pi\left(\frac{\sin\Phi_2}{1-e^2\cdot\sin^2\Phi_2} - \frac{\sin\Phi_1}{1-e^2\cdot\sin^2\Phi_1} + \frac{1}{2e}\cdot\ln\frac{1+e\cdot\sin\Phi_2}{1-e\cdot\sin\Phi_2} - \frac{1}{2e}\cdot\ln\frac{1+e\cdot\sin\Phi_1}{1-e\cdot\sin\Phi_1}\right)\cdot\frac{arc(\Lambda_2-\Lambda_1)}{2\pi}$$

- The surface area of the *whole ellipsoid*:

$$F_e = 2 \cdot a^2 \cdot \pi \cdot \left[1 + \frac{(1-e^2)}{2 \cdot e} \cdot ln\frac{1+e}{1-e}\right]$$

- The surface area F_l of an *ellipsoidal lune* between the longitudes Λ_1 and Λ_2 is its proportional part:

$$F_1 = a^2 \cdot \left[1 + \frac{(1-e^2)}{2 \cdot e} \cdot ln\frac{1+e}{1-e}\right] \cdot arc(\Lambda_2 - \Lambda_1)$$

- **Example**: Calculate the surface area of the WGS84 ellipsoid, the zone between the Equator and the parallel of the Bp point and the lune between the Greenwich meridian and the meridian of the Bp point.

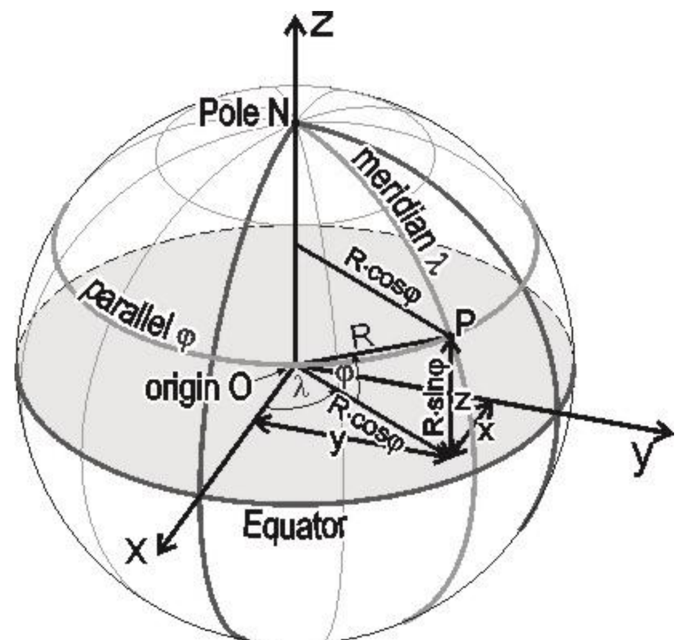
Transformations among the different types of coordinate systems on the spherical surfaces

Transformation between geographic and spatial rectangular coordinates

The origin of the two coordinate systems as well as the axis *z* and the polar axis concide, then the axis x is in the prime semi-plane, additionally the units at the axes are the same.

forward formulae:

$$x = R \cdot \cos\varphi \cdot \cos\lambda$$
$$y = R \cdot \cos\varphi \cdot \sin\lambda$$
$$z = R \cdot \sin\varphi$$



reverse formulae:

$$\varphi = \arcsin\left(\frac{z}{R}\right) = \arccos\left(\frac{\sqrt{x^2 + y^2}}{R}\right)$$

$$\lambda = \arctan\left(\frac{y}{x}\right) = \arccos\left(\frac{x}{R \cdot \cos\varphi}\right) = \arcsin\left(\frac{y}{R \cdot \cos\varphi}\right)$$

Transformations among the different types of coordinate systems on ellipsoidal surfaces (1)

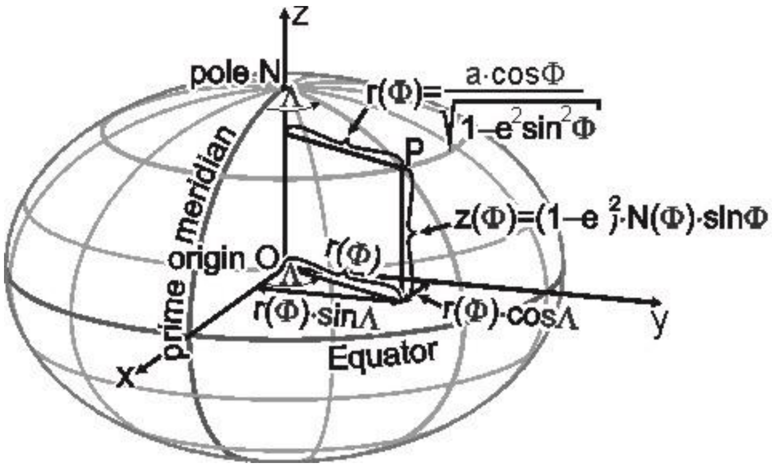
- **Formulae disregarding the elevation**

<u>forward formulae</u>:

$$X = r(\Phi)\cdot\cos\Lambda = N(\Phi)\cdot\cos\Phi\cdot\cos\Lambda = \frac{a\cdot\cos\Phi}{\sqrt{1-e^2\cdot\sin^2\Phi}}\cdot\cos\Lambda$$

$$Y = r(\Phi)\cdot\sin\Lambda = N(\Phi)\cdot\cos\Phi\cdot\sin\Lambda = \frac{a\cdot\cos\Phi}{\sqrt{1-e^2\cdot\sin^2\Phi}}\cdot\sin\Lambda$$

$$Z = \left(1-e^2\right)\cdot N(\Phi)\cdot\sin\Phi = \left(1-e^2\right)\cdot\frac{a\cdot\sin\Phi}{\sqrt{1-e^2\cdot\sin^2\Phi}}$$



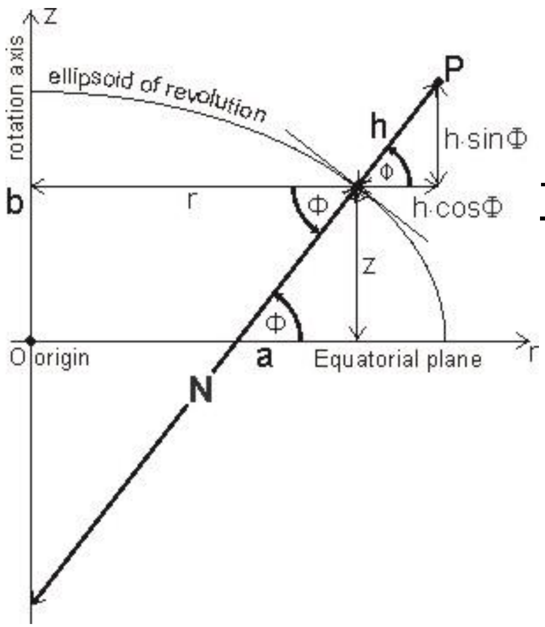
<u>reverse formulae</u>: Φ can be expressed from the formula for Z by rearranging it, with respect to $\sin\Phi$:

$$\Phi = \arcsin\left(\frac{Z}{\sqrt{\left(1-e^2\right)^2\cdot a^2+e^2\cdot Z^2}}\right) = \arcsin\left(\frac{a\cdot Z}{\sqrt{b^4+\left(a^2-b^2\right)\cdot Z^2}}\right)$$

$$\Lambda = \arctan\left(\frac{Y}{X}\right)$$

Transformations among the different types of coordinate systems on ellipsoidal surfaces (2)

- **Formulae taking elevation *h* into consideration**



Note: In the right triangle the length of the hypotenuse equals:

$$\frac{r}{\cos\Phi} = \frac{a\cdot\cos\Phi}{\cos\Phi\cdot\sqrt{1-e^2\cdot\sin^2\Phi}} = N$$

<u>forward formulae</u>:

$$X = [r(\Phi)+h\cdot\cos\Phi]\cdot\cos\Lambda = [N(\Phi)+h]\cdot\cos\Phi\cdot\cos\Lambda$$

$$Y = [r(\Phi)+h\cdot\cos\Phi]\cdot\sin\Lambda = [N(\Phi)+h]\cdot\cos\Phi\cdot\sin\Lambda$$

$$Z = \frac{b^2}{a^2}\cdot N(\Phi)\cdot\sin\Phi + h\cdot\sin\Phi = [(1-e^2)\cdot N(\Phi)+h]\cdot\sin\Phi$$

The reverse formulae providing the (Φ, Λ, h) from the spatial rectangular coordinates (X, Y, Z) are important for the calculations of the GPS mesurements (see later).

Transformations among the different types of coordinate systems on ellipsoidal surfaces (3)

- reverse formulae by iteration (Bowring)

initial values $\bar{\varphi}$ and $\bar{\Phi}$ for the reduced latitude φ and geographic latitude Φ:

$$\bar{\varphi} = \arctan\left(\frac{b}{a} \cdot \frac{Z}{\sqrt{X^2+Y^2}}\right) \cdot \left(1 + \frac{b \cdot (e')^2}{\sqrt{X^2+Y^2+Z^2}}\right)$$

and

$$\bar{\Phi} = \arctan\left(\frac{Z + b \cdot (e')^2 \cdot \sin^3 \bar{\varphi}}{\sqrt{X^2+Y^2} - a \cdot e^2 \cdot \cos^3 \bar{\varphi}}\right)$$

then the next three joint formulae:

$$\Phi = \bar{\Phi}$$

$$\varphi = \arctan\left(\frac{b}{a} \cdot \tan\Phi\right)$$

$$\bar{\Phi} = \arctan\left(\frac{Z + (e')^2 \cdot b \cdot \sin^3 \varphi}{\sqrt{X^2+Y^2} - a \cdot e^2 \cdot \cos^3 \varphi}\right)$$

should be executed repeatedly one after the other, until the deviation between the two last latitude values is smaller than the required accuracy. Thus receiving Φ, the formula for the elevation *h* from Z is:

$$h = \frac{Z}{\sin\Phi} - \left(1 - e^2\right) \cdot N(\Phi)$$

Finally the longitude is as usual: $\Lambda = \arctan\left(\frac{Y}{X}\right)$

Transformations among the different types of coordinate systems on ellipsoidal surfaces (4)

<u>reverse formulae</u> by accurate algorithm (Borkowski):

$\bar{r}$ denotes the distance of the point P from the ellipsoid's rotation axis:

$$\bar{r} = \sqrt{X^2 + Y^2} = [N(\Phi) + h] \cdot \cos\Phi \cdot \sqrt{\cos^2\Lambda + \sin^2\Lambda}$$

h is expressed using both this formula and the formula for *Z*

$$\frac{\bar{r}}{\cos\Phi} - N(\Phi) = \frac{Z}{\sin\Phi} - (1 - e^2) \cdot N(\Phi) \qquad \text{that is} \qquad \frac{\bar{r}}{\cos\Phi} - \frac{Z}{\sin\Phi} = \frac{e^2 \cdot a}{\sqrt{1 - e^2 \sin^2\Phi}}$$

then squaring and multiplying by the denominators:

$$\bar{r}^2 \cdot \sin^2\Phi - 2Z \cdot \bar{r} \cdot \sin\Phi \cdot \cos\Phi + Z^2 \cdot \cos^2\Phi - \bar{r}^2 \cdot e^2 \cdot \sin^4\Phi +$$
$$+ 2 \cdot Z \cdot \bar{r} \cdot e^2 \cdot \sin^3\Phi \cdot \cos\Phi - Z^2 \cdot e^2 \cdot \cos^2\Phi \cdot \sin^2\Phi = a^2 \cdot e^4 \cdot \sin^2\Phi \cdot \cos^2\Phi$$

rearranging the equation, dividing by $\cos^4\Phi$, applying the identity $\frac{1}{\cos^2\Phi} = 1 + \tan^2\Phi$: a *quartic equation* of $\tan\Phi$ will be got:

$$\tan^4\Phi \cdot \bar{r}^2 \cdot (1 - e^2) + \tan^3\Phi \cdot 2 \cdot Z \cdot \bar{r} \cdot (e^2 - 1) +$$
$$+ \tan^2\Phi \cdot \left[\bar{r}^2 + Z^2 \cdot (1 - e^2) - a^2 \cdot e^4\right] + \tan\Phi \cdot (-2 \cdot \bar{r} \cdot Z) + Z^2 = 0$$

It can be solved exactly e.g. by the Ferrari's method.

- The longitude Λ and the elevation *h* can be got similarly to the Bowring method: $h = \frac{Z}{\sin\Phi} - (1 - e^2) \cdot N(\Phi)$ and $\Lambda = \arctan\left(\frac{Y}{X}\right)$

Examples:

Let the Bp point coordinates be considered as spherical ones:

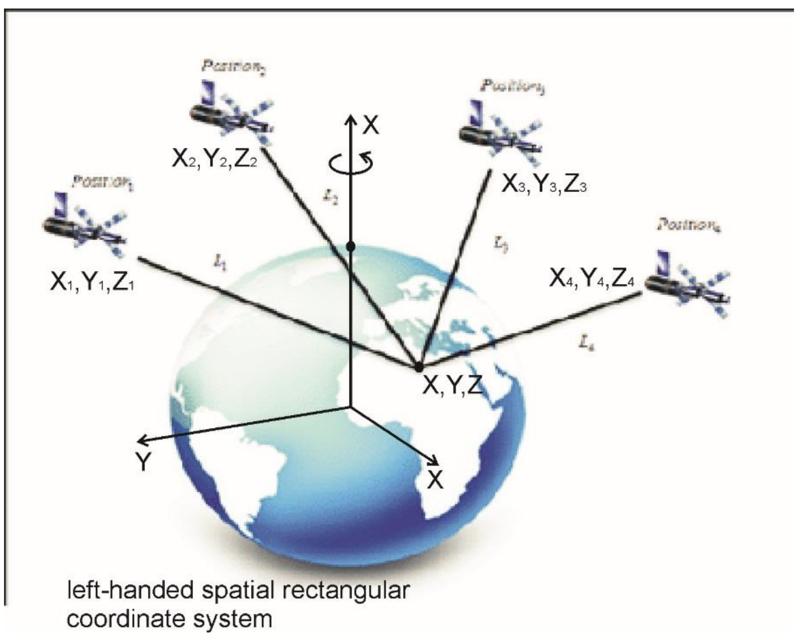
latitude φ = 47°28'29.262"

longitude λ = 19°3'43.303"

- Calculate the spatial rectangular coordinates of this point
- Calculate the latitude and longitude from the spatial rectangular coordinates
- Let the geographic coordinates above considered as WGS84 ones, and the elevation above the ellipsoid: h=187.575 m. Calculate the spatial rectangular coordinates of this point with elevation.
- Calculate the latitude, longitude and elevation from the spatial rectangular coordinates by the Bowring method.
- Calculate the surface area of sphere (R=6371 km) and the WGS84 ellipsoid
- **Home work:**
- Surface area of the geographic quadrangle bounded by the Equator and the parallel of the GPS point, as well as the Greenwich meridian and the meridian of the GPS point, both on the sphere and on the WGS84 ellipsoid =?
- Let the geographic coordinates above considered as WGS84 ones. Calculate the spatial rectangular coordinates of this point without elevation
- Calculate the latitude and longitude from the spatial rectangular coordinates

The working principle of the GPS (1)

- A satellite system is composed of 29 satellites circulating on definite orbits. Their positions (X_i, Y_i, Z_i) i=1,2,…,29 at a specific time instant are known from the orbit data.
- Every satellite has a highly exact atomic clock, and the GPS has a clock.
- The **time difference** which passes from emitting the signal of the satellite i to receiving the signal by the GPS will be converted into the distance L_i with the help of the propagation speed of the radio waves.



The working principle of the GPS (2)

- If **three satellites** are visible for the GPS then the formulae for the distances L_1, L_2, L_3 from the coordinate differences (Pythagorean theorem):

$$L_1 = \sqrt{(X - X_1)^2 + (Y - Y_1)^2 + (Z - Z_1)^2}$$
$$L_2 = \sqrt{(X - X_2)^2 + (Y - Y_2)^2 + (Z - Z_2)^2}$$
$$L_3 = \sqrt{(X - X_3)^2 + (Y - Y_3)^2 + (Z - Z_3)^2}$$

  where (*X, Y, Z*) are the spatial rectangular coordinates of the position of the GPS. This is a non-linear system of equations with respect to (*X, Y, Z*). It can be solved by iterative methods (e.g. Newton-Raphson).

- The clock in the GPS is not so accurate as the atomic clocks of the satellites. Therefore the measuring of the time differences has an error. It will be converted into a distance error ΔL which can be added to the distances originated in the coordinate differences.

The working principle of the GPS (3)

- If **four satellites** are visible then the system of equations consists of four equations and the ΔL can be taken into consideration as a fourth unknown variable outside of (X,Y,Z):

$$L_1 = \sqrt{(X - X_1)^2 + (Y - Y_1)^2 + (Z - Z_1)^2} + \Delta L$$

$$L_2 = \sqrt{(X - X_2)^2 + (Y - Y_2)^2 + (Z - Z_2)^2} + \Delta L$$

$$L_3 = \sqrt{(X - X_3)^2 + (Y - Y_3)^2 + (Z - Z_3)^2} + \Delta L$$

$$L_4 = \sqrt{(X - X_4)^2 + (Y - Y_4)^2 + (Z - Z_4)^2} + \Delta L$$

- After solving this system, the obtained coordinates (*X, Y, Z*) are more exact than the ones originating in the previous system with three equations.
- A further calculation is needed for the geographic coordinates (Φ, Λ) and the elevation *h* (namely Φ from the Bowring's iterative method or Borkowski's accurate algorithm, then *h* and Λ from the formulae above).

Examples (Lesson_3):

- Budapest (Control point 1) WGS84 coordinates:

  latitude Φ = 47°28’29.262” (radian=?)

  longitude Λ = 19°3’43.303” (radian=?)
- Length Δm of the meridian arc between the Equator and the Budapest point on the WGS84 ellisoid =?
- Length Δp of the parallel arc between the Bp point and the Greenwich prime meridian =?
- Distance Δz between the parallel crossing Bp point and the Equator =?
- Extra credit example: the (perpendicular) distance of the Bp point from the Greenwich prime semi-plane =?
- The length of the boundaries and the ellipsoidal surface area of Colorado state (US) =? (37°<=Φ<=41°, –109°02’<=Λ<=–102°03’)

  GRS80 ellipsoid (a = 6378137m, 1/f = 298.257222101)