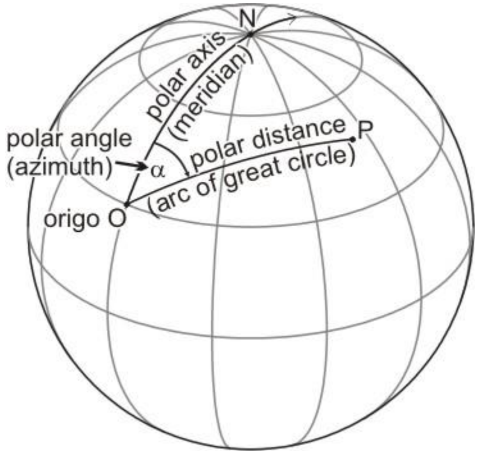
Lesson 4

- The basics of the spherical trigonometry
- Transformations between the superficial polar and geographic coordinates
- The metacoordinate system and its transformation into geographic coordinate system back and forth

Superficial polar coordinate system on spherical surface

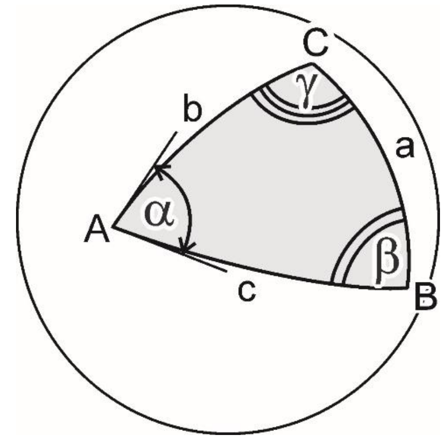


- *Origin* O is appointed on a meridian.
- *Polar axis*: coincides with a part of a spherical meridian (great circle arc, geodesic line), starting from the origin O.
- *Polar distance*: length of the great circle arc (geodesic line) connecting the origin O and the point P.
- *Polar angle* (azimut) can be measured at the origin O between the tangent lines of the meridian and the great circle arc, directed clockwise.
- Problems connected to these coordinates lead up to calculations of *spherical trigonometry*.

The basics of the spherical trigonometry

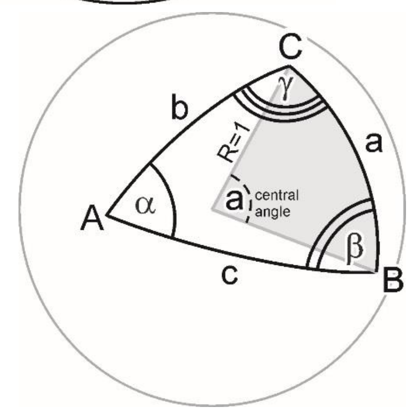
Spherical triangle: bounded by three great circle arcs (geodetic lines)

- 3 *vertices* (denoted by great Latin letters)
- 3 *sides* (denoted by small Latin letters)
- 3 *angles* (denoted by small Greek letters) defined by the angle between the tangent lines of two sides (crossing a vertex)



Some fundamental rules for the spherical triangles:

- greater angle is opposite to greater side (if a>b then $\alpha>\beta$);
- equal angles are opposite to equal sides (if a=b then $\alpha=\beta$);
- the sum of the lengths of two sides is greater than the length of the third side (e.g. a+b>c);
- the sum of the angles of a real (not degenerated) spherical triangle is greater than the straight angle ($\alpha+\beta+\gamma>180°$)



$\varepsilon=\alpha+\beta+\gamma-180°$ spherical excess; *surface area* of a spherical triangle: $F=R^2\cdot\mathrm{arc}\,\varepsilon$

The spherical triangle can be symmetrical, regular, right triangle, too.

From now on only *convex* spherical triangles will be used!

On a *unit sphere* (R=1) the radian measure of the central angle equals the arc length of the side. Conventionally *the sides* of a spherical triangle *are given by* the correspondent *central angles*.

Connections among the data (sides and angles) of the spherical triangle

Three of the six data of a general spherical triangle (all given by angles) are independent determining the others.

- **Law of sines** (sine rule) $$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}$$

  (connections between the sides and the opposite angles)

- **Cosine rule for sides** (first cosine rule)

$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos \alpha$$
$$\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos \beta$$
$$\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos \gamma$$

  (connections among the three sides and one angle)

- **Cosine rule fos angles** (second cosine rule)

$$\cos \alpha = -\cos \beta \cdot \cos \gamma + \sin \beta \cdot \sin \gamma \cdot \cos a$$
$$\cos \beta = -\cos \alpha \cdot \cos \gamma + \sin \alpha \cdot \sin \gamma \cdot \cos b$$
$$\cos \gamma = -\cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta \cdot \cos c$$

  (connections among the three angles and one side)

Application of the connections related to the spherical triangle

The basic cases for calculating the missing data of a spherical triangle:

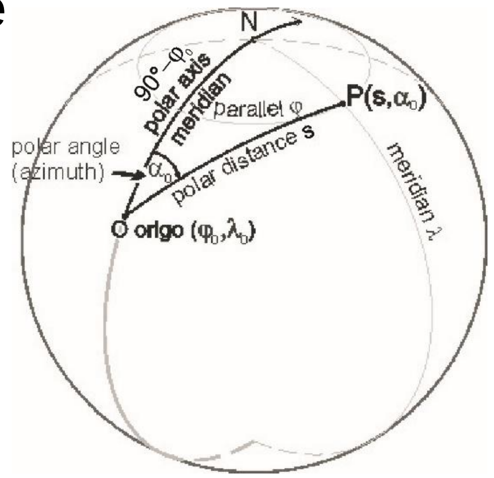
- given two sides and the angle between them (start with the suitable cosine rule for sides);
- given one side and the two angles on it (start with the suitable cosine rule for angles);
- given three sides (start with the suitably rearranged cosine rule for sides);
- given three angles (start with the suitably rearranged cosine rule for angles);

The law of sines generally can not be applied on its own, but it is appropriate for the further calculations

- **Example**:
- Given: a=60°, b=90°, γ=45°. Asked: c=? α=? β=? F=?

First (direct) geodetic problem (1)

- **Origin O** is appointed with its geographic coordinates (φ_0, λ_0)
- **Given** the polar distance (arc length *s*) of the point P from the origin O (φ_0, λ_0) and its azimuth α_0 on the spherical surface
- **Asked** the geographic coordinates φ_P and λ_P of P



- **Latitude φ_P**: starting from the **spherical triangle NOP** (where N is the pole), the *cosine rule for sides* can be applied:

$$\cos(90° - \varphi_P) = \cos(90° - \varphi_0) \cdot \cos\left(\frac{s}{R}\right) + \sin(90° - \varphi_0) \cdot \sin\left(\frac{s}{R}\right) \cdot \cos\alpha_0$$

(Note: s/R is the central angle in radian mesure belongig to the arc s in the great circle with radius R).

- Expressing the latitude φ_P:

$$\varphi_P = \arcsin\left[\sin\varphi_0 \cdot \cos\left(\frac{s}{R}\right) + \cos\varphi_0 \cdot \sin\left(\frac{s}{R}\right) \cdot \cos\alpha_0\right]$$

First (direct) geodetic problem (2)

- **Longitude difference** $\Delta\lambda = \lambda_P - \lambda_0$ is resulted in by the *law of sines*

$$\frac{\sin(\Delta\lambda)}{\sin\left(\frac{s}{R}\right)} = \frac{\sin\alpha_0}{\sin(90^\circ - \varphi_P)} = \frac{\sin\alpha_0}{\cos\varphi_P}$$

and by the *cosine rule for sides*:

$$\cos\left(\frac{s}{R}\right) = \cos(90^\circ - \varphi_0)\cdot\cos(90^\circ - \varphi_P) + \sin(90^\circ - \varphi_0)\cdot\sin(90^\circ - \varphi_P)\cdot\cos\Delta\lambda$$

The modulus of $\Delta\lambda$ originates in the $\cos(\Delta\lambda)$, and its sign in the $\sin(\Delta\lambda)$.

The longitude difference $\Delta\lambda$:

$$\Delta\lambda = \arccos\left[\frac{\cos\left(\frac{s}{R}\right) - \sin\varphi_0\cdot\sin\varphi_P}{\cos\varphi_0\cdot\cos\varphi_P}\right]\cdot\frac{\sin\Delta\lambda}{|\sin\Delta\lambda|}$$

Finally

$$\lambda_P = \lambda_0 + \Delta\lambda$$

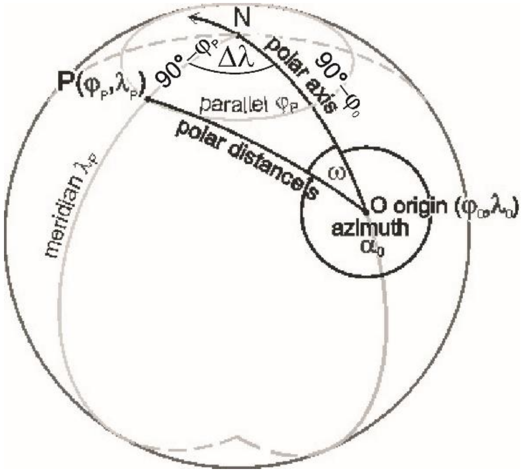
Example

- Given: coordinates of the origin O (φ_0=47.5°, λ_0=19°), the polar distance of the point P (s=1437553.527 m) and the azimuth (polar angle) of the point P: (α_0 =295.059088°)
- Asked: the geographic coordinates of the point P (φ_0, λ_0).

Second (inverse) geodetic problem (1)

- **Given:** the geographic coordinates φ_P and λ_P of the point P on spherical surface;

  Asked: the polar distance (arc length *s*) of the point P from the origin $O(\varphi_0, \lambda_0)$ and its azimuth α_0.



Polar distance (arc length *s*): starting from the **spherical triangle NOP** (where N is the pole), the *cosine rule for sides* can be applied:

$$\cos\left(\frac{s}{R}\right) = \cos(90° - \varphi_0)\cdot\cos(90° - \varphi_P) + \sin(90° - \varphi_0)\cdot\sin(90° - \varphi_P)\cdot\cos\Delta\lambda$$

where $\Delta\lambda = \lambda_P - \lambda_0$

The **arc length *s*** can be expressed

$$s = R \cdot arccos[sin(\varphi_0) \cdot sin(\varphi_P) + cos(\varphi_0) \cdot cos(\varphi_P) \cdot cos\Delta(\lambda)]$$

Note. The angle resulting in *arccos* of the formula above is considered as given in radian measure otherwise it has to be converted.

Second (inverse) geodetic problem (2)

- **Azimuth α_0 :** the interior angle ω lying next to the point O of the **spherical triangle NOP** can be calculated by the *law of sines:*

$$\sin\omega = \frac{\sin(90° - \varphi_P)\cdot\sin(\Delta\lambda)}{\sin\left(\frac{s}{R}\right)}$$

and by the rearranged *cosine rule for sides*, too:

$$\cos\omega = \frac{\cos(90° - \varphi_P) - \cos(90° - \varphi_0)\cdot\cos\left(\frac{s}{R}\right)}{\sin(90° - \varphi_0)\cdot\sin\left(\frac{s}{R}\right)} = \frac{\sin\varphi_P - \sin\varphi_0\cdot\cos\left(\frac{s}{R}\right)}{\cos\varphi_0\cdot\sin\left(\frac{s}{R}\right)}$$

Thus

$$\omega = arccos\left[\frac{sin\varphi_P - sin\varphi_0\cdot cos\left(\frac{s}{R}\right)}{cos\varphi_0\cdot sin\left(\frac{s}{R}\right)}\right]\cdot\frac{sin\omega}{|sin\omega|}$$

If the point P is located to east from the meridian of the origin O, then

$$\alpha_0 = \omega$$

else

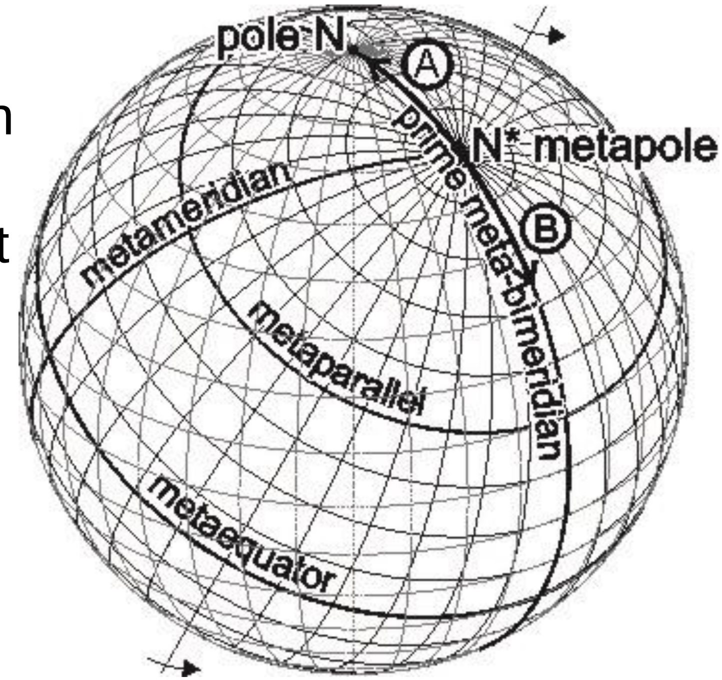
$$\alpha_0 = 360° - \omega$$

Metacoordinates, metacoordinate system

- On the *spherical* surface, a new reference system, *the metacoordinate system,* can be established by the rotation(s) of the geographic coordinate system about axes containing the centre of the sphere. The main purpose of its application is to discuss pro forma different map projections *uniformly*.

  It can not be defined on ellipsoidal surfaces.
- **Appointment of the metapole N*** generates *metalatitudes and metaparallels* furthermore an axis crossing N* and the centre of the sphere;
- The semi-planes bounded by this axis intersect the spherical surface in half great circles.

  After appointing the *prime semi-plane*, the angle between the prime semi-plane and certain such semi-planes provides the *metalongitude*, the intersection lines of these semi-planes and the spherical surface give the *metameridians*. The prime metameridian generally crosses one of the poles.

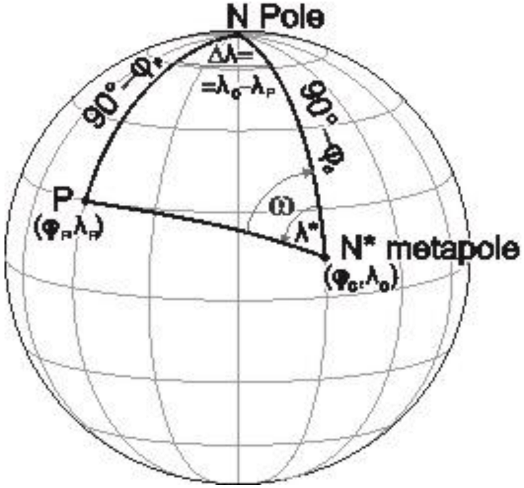


Transformations between geographic coordinates and metacoordinates using the metapole (1)

- **The metapole N***(φ_0, λ_0) **and the prime metameridian** are appointed (especially in the case of points *around the metapole*)
- **Given:** the coordinates (φ_P, λ_P) of the point P;

  Asked: the metacoordinates (φ^*, λ^*) of P.

  Choosing N* as origin, it is an *inverse geodetic problem*. Let the prime meta-meridian as polar axis be directed from N* e.g. towards the pole N.



The **metalatitude** φ^* is the complementary angle of the central angle of N*P:

$$\varphi^* = \arcsin\left(\sin\varphi_0 \cdot \sin\varphi_P + \cos\varphi_0 \cdot \cos\varphi_P \cdot \cos(\Delta\lambda)\right)$$ where $\Delta\lambda = \lambda_P - \lambda_0$

The **metalongitude** λ^* is the angle beside N*:

$$\lambda^* = \arccos\left[\frac{\sin\varphi_P - \sin\varphi_0 \cdot \sin\varphi^*}{\cos\varphi_0 \cdot \cos\varphi^*}\right] \cdot \frac{(-\sin\Delta\lambda)}{|\sin\Delta\lambda|}$$

Transformations between geographic coordinates and metacoordinates using the metapole (2)

- **Given**: the metacoordinates (φ^*,λ^*) of the point P.
  Asked: the coordinates (φ_P , λ_P) of the point P;
  Choosing N* as origin, it is a *direct geodetic problem*.

The **latitude** φ_P is the complementary angle of the central angle of N*P:

$$\varphi_P = \arcsin\left(\sin\varphi_0 \cdot \sin\varphi^* + \cos\varphi_0 \cdot \cos\varphi^* \cdot \cos\lambda^*\right)$$

The **longitude** λ_P comes from the longitude difference $\Delta\lambda$:

$$\Delta\lambda = \arccos\left[\frac{\sin\varphi^* - \sin\varphi_0 \cdot \sin\varphi_P}{\cos\varphi_0 \cdot \cos\varphi_P}\right] \cdot \frac{-\sin\lambda^*}{\left|\sin\lambda^*\right|}$$

and $\lambda_P = \lambda_0 + \Delta\lambda$

Discussion: (zero cannot be in the denominator of λ): .

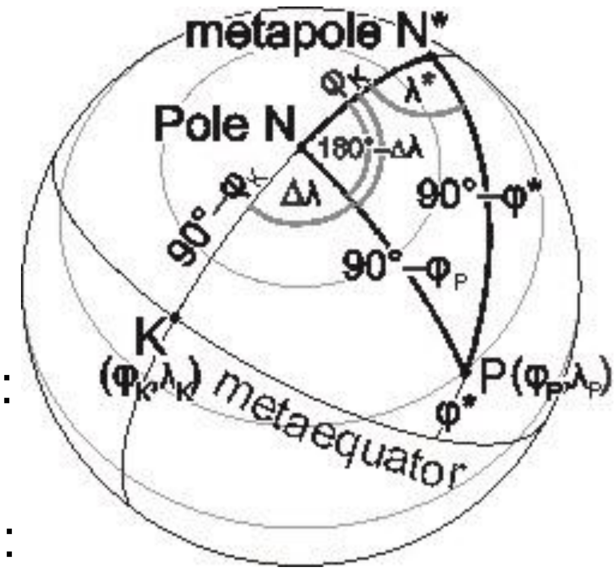
- If cos φ_0 = 0 then φ_0 = $\pm$90° that is $\varphi^* = \pm\varphi$ and $\lambda^* \pm \lambda$+const
- If cos φ_P = 0 then φ_P = $\pm$90° that is P coincides with the pole, then its longitude is not interpreted, or in other words its longitude can be arbitrary.
- If sin(λ^*)=0 then P is on the bimeridian of N* and N, so: $\Delta\lambda$=0 or $\Delta\lambda$=$\pm$180°

Transformations between geographic coordinates and metacoordinates using the intersection point of the metaequator and the prime metameridian (1)

- The **intersection point K**(φ_K ,λ_K) of the metaequator and the prime meta-meridian is appointed (used in the case of points around the metaequator)
- **Given**: the coordinates (φ_P ,λ_P) of the point P;

  Asked: the metacoordinates (φ^*,λ^*).

  The metapole N* considered as known based on K. In the spherical triangle NN*P the side PN* gives the complementary angle of the **metalatitude** φ^*. From the *cosine rule for sides* :



$$\varphi^* = \arcsin\left(\cos\varphi_K \cdot \sin\varphi_P - \sin\varphi_K \cdot \cos\varphi_P \cdot \cos(\Delta\lambda)\right)$$

($\Delta\lambda=\lambda_P-\lambda_K$); The **metalongitude** λ^*: by *sine rule*:

$$\lambda^* = \arcsin\left(\frac{\cos\varphi_P \cdot \sin\Delta\lambda}{\cos\varphi^*}\right) = \arcsin\left(\frac{\cos\varphi_P \cdot \sin\Delta\lambda}{\sqrt{1-(\cos\varphi_K \cdot \sin\varphi_P - \sin\varphi_K \cdot \cos\varphi_P \cdot \cos\Delta\lambda)^2}}\right)$$

or cos λ^* can be expressed from *the cosine rule for sides* related to PN,

then $\tan\lambda^* = \dfrac{\sin\lambda^*}{\cos\lambda^*}$ and it follows $\lambda^* = \arctan\left(\dfrac{\sin\Delta\lambda}{\sin\varphi_K \cdot \tan\varphi_P + \cos\varphi_k \cdot \cos\Delta\lambda}\right)$

Transformations between geographic coordinates and metacoordinates using the intersection point of the metaequator and the prime metameridian (2)

- **Given**: the metacoordinates (φ^*,λ^*) of the point P ;
  Asked: the coordinates (φ_P , λ_P) of the point P.

  The **latitude** φ_P : by the *cosine rule for sides*

  $$\varphi_P = \arcsin\left(\cos\varphi_K \cdot \sin\varphi^* + \sin\varphi_K \cdot \cos\varphi^* \cdot \cos\lambda^*\right)$$

  The **longitude** λ_P : by the *law of sines*

  $$\Delta\lambda = \arcsin\left(\frac{\cos\varphi^* \cdot \sin\lambda^*}{\sqrt{1-\left(\cos\varphi_K \cdot \sin\varphi^* + \sin\varphi_K \cdot \cos\varphi^* \cdot \cos\lambda^*\right)^2}}\right)$$

  or cos($\Delta\lambda$) can be expressed from *the cosine rule for sides* related to PN*,

  then $\tan\Delta\lambda = \dfrac{\sin\Delta\lambda}{\cos\Delta\lambda}$ and it follows $\Delta\lambda = \arctan\left(\dfrac{\sin\lambda^*}{\cos\varphi_K \cdot \cos\lambda^* - \sin\varphi_k \cdot \tan\varphi^*}\right)$

  finally $\lambda_P = \lambda_K + \Delta\lambda$

  Example: Given: the origin K (φ_K=47.5°,λ_K=19°), the point P (Rome): (φ_P=41.893117°, λ_P=12.484917°).
  Asked: the metacoordinates (φ^*,λ^*) of the point P.

Example

- Given: coordinates of the origin O (φ_0=47.5°, λ_0=19°), the coordinates of the point P (φ_P=47.5°, λ_P=19°)

  Asked: the polar distance *s* and the azimuth α_0 (polar angle) of the point P

Home work:

1. Given: the sides of a spherical triangle (a=60°, b=75°, c=90°).

Asked: its angles α, β, γ and its surface area F.

2. Given: coordinates of the origin O (φ_0 =47.5°, λ_0 =19°), and the geographic coordinates of a GPS point;

Asked: the polar distance from the origin O and the azimuth α_0 (polar angle) of the GPS point.

3. Given: the polar distance from the origin O and the azimuth α_0 (polar angle) of the GPS point originating from 2.;

Asked: the geographic coordinates of the GPS point.