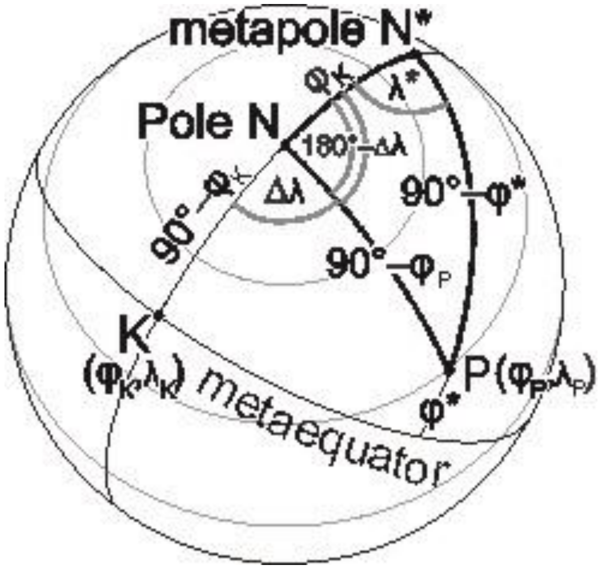
Lesson 5

- Supplement: transformation between geographic coordinate and metacoordinate systems back and forth using the intersection point of the metaequator and the prime metameridian
- Geodetic datums
- Transformations of geographic coordinates between different reference surfaces

Transformations between geographic coordinates and metacoordinates using the intersection point of the metaequator and the prime metameridian (1)

- The **intersection point K**(φ_K , λ_K) of the metapole and the prime meta-meridian is appointed (used in the case of points around the metaequator)
- **Given**: the coordinates (φ_P , λ_P) of the point P;

  Asked: the metacoordinates (φ^*,λ^*).



  The metapole N* considered known based on K. In the spherical triangle NN*P the side PN* gives the complementary angle of the **metalatitude** φ^*. From the *cosine rule for sides* :

$$\varphi^* = \arcsin\left(\cos\varphi_K \cdot \sin\varphi_P - \sin\varphi_K \cdot \cos\varphi_P \cdot \cos(\Delta\lambda)\right)$$

  ($\Delta\lambda=\lambda_P-\lambda_K$); The **metalongitude** λ^*: by *sine rule*:

$$\lambda^* = \arcsin\left(\frac{\cos\varphi_P \cdot \sin\Delta\lambda}{\cos\varphi^*}\right) = \arcsin\left(\frac{\cos\varphi_P \cdot \sin\Delta\lambda}{\sqrt{1-\left(\cos\varphi_K \cdot \sin\varphi_P - \sin\varphi_K \cdot \cos\varphi_P \cdot \cos\Delta\lambda\right)^2}}\right)$$

  or cos λ^* can be expressed from *the cosine rule for sides* related to PN,

  then $\tan\lambda^* = \dfrac{\sin\lambda^*}{\cos\lambda^*}$ and it follows $\lambda^* = \arctan\left(\dfrac{\sin\Delta\lambda}{\sin\varphi_K \cdot \tan\varphi_P + \cos\varphi_k \cdot \cos\Delta\lambda}\right)$

Transformations between geographic coordinates and metacoordinates using the intersection point of the metaequator and the prime metameridian (2)

- **Given**: the metacoordinates (φ^*,λ^*) of the point P ;
  Asked: the coordinates (φ_P , λ_P) of the point P.

  The **latitude** φ_P : by the *cosine rule for sides*

  $$\varphi = \arcsin\left(\cos\varphi_K \cdot \sin\varphi^* + \sin\varphi_K \cdot \cos\varphi^* \cdot \cos\lambda^*\right)$$

  The **longitude** λ_P : by the *law of sines*

  $$\Delta\lambda = \arcsin\left(\frac{\cos\varphi^* \cdot \sin\lambda^*}{\sqrt{1-\left(\cos\varphi_K \cdot \sin\varphi^* + \sin\varphi_K \cdot \cos\varphi^* \cdot \cos\lambda^*\right)^2}}\right)$$

  or cos($\Delta\lambda$) can be expressed from *the cosine rule for sides* related to PN*,

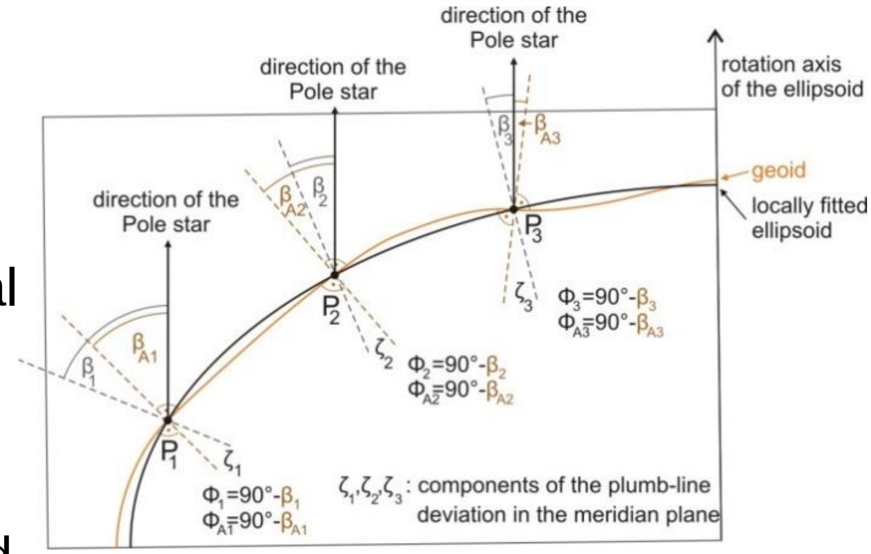
  then $\tan\Delta\lambda = \dfrac{\sin\Delta\lambda}{\cos\Delta\lambda}$ and it follows $\Delta\lambda = \arctan\left(\dfrac{\sin\lambda^*}{\cos\varphi_K \cdot \cos\lambda^* - \sin\varphi_k \cdot \tan\varphi^*}\right)$

  finally $\lambda_P = \lambda_K + \Delta\lambda$

  (φ_P=41.893117°, λ_P=12.484917°).

Fitting the ellipsoid to the geoid. Geodetic datums

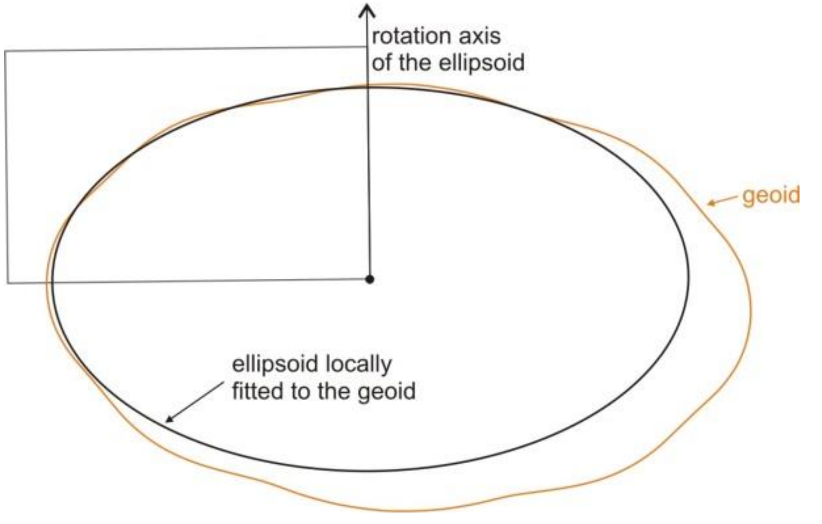
- The astronomical coordinates of a point on the Earth surface are unambiguously defined on the geoid. The ellipsoidal geographic coordinates of the same point depend *on the position of the ellipsoid related to the geoid*. To their unambiguity, the ellipsoid has to be fitted to the geoid.
- Laplace-points: triangulation points where both geodetic and astronomic measurements were established. Thus the deviation of the local vertical („plumb-line deviation” or „deflection of the vertical”), related both to ellipsoid and geoid, can be determined.



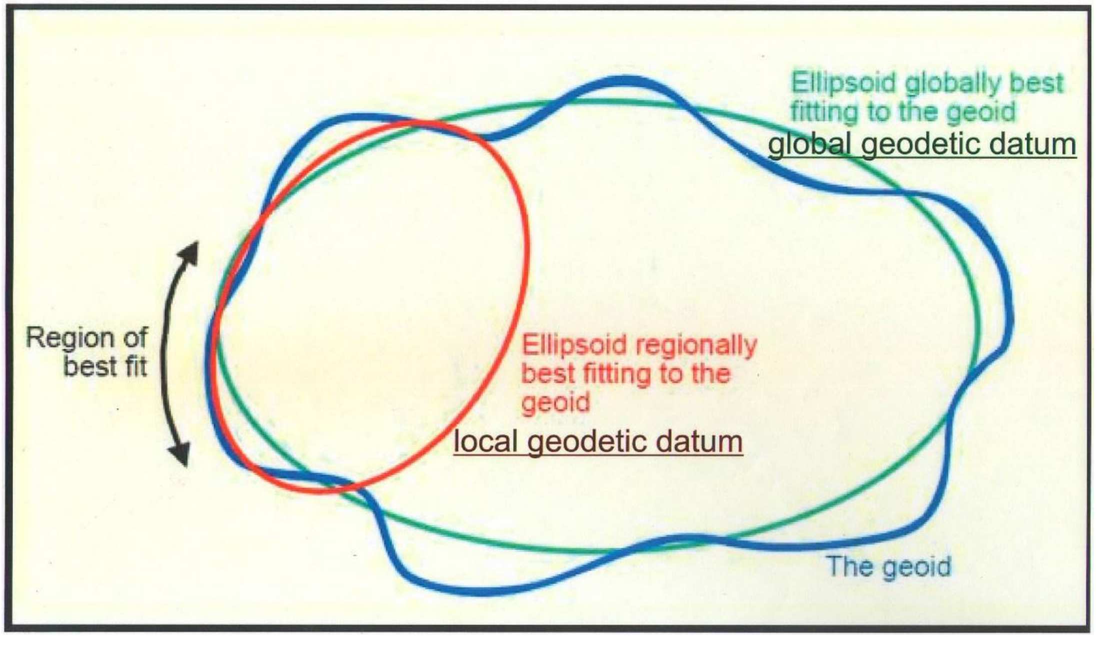
- The ellipsoid will be fitted to the geoid by the minimisation of the sum of the squared angle differences between the vertical directions on the geoid and the ellipsoid
- This fitted ellipsoidal surface is called the *geodetic datum* (or simply *datum*).

Locally and globally fitted ellipsoids

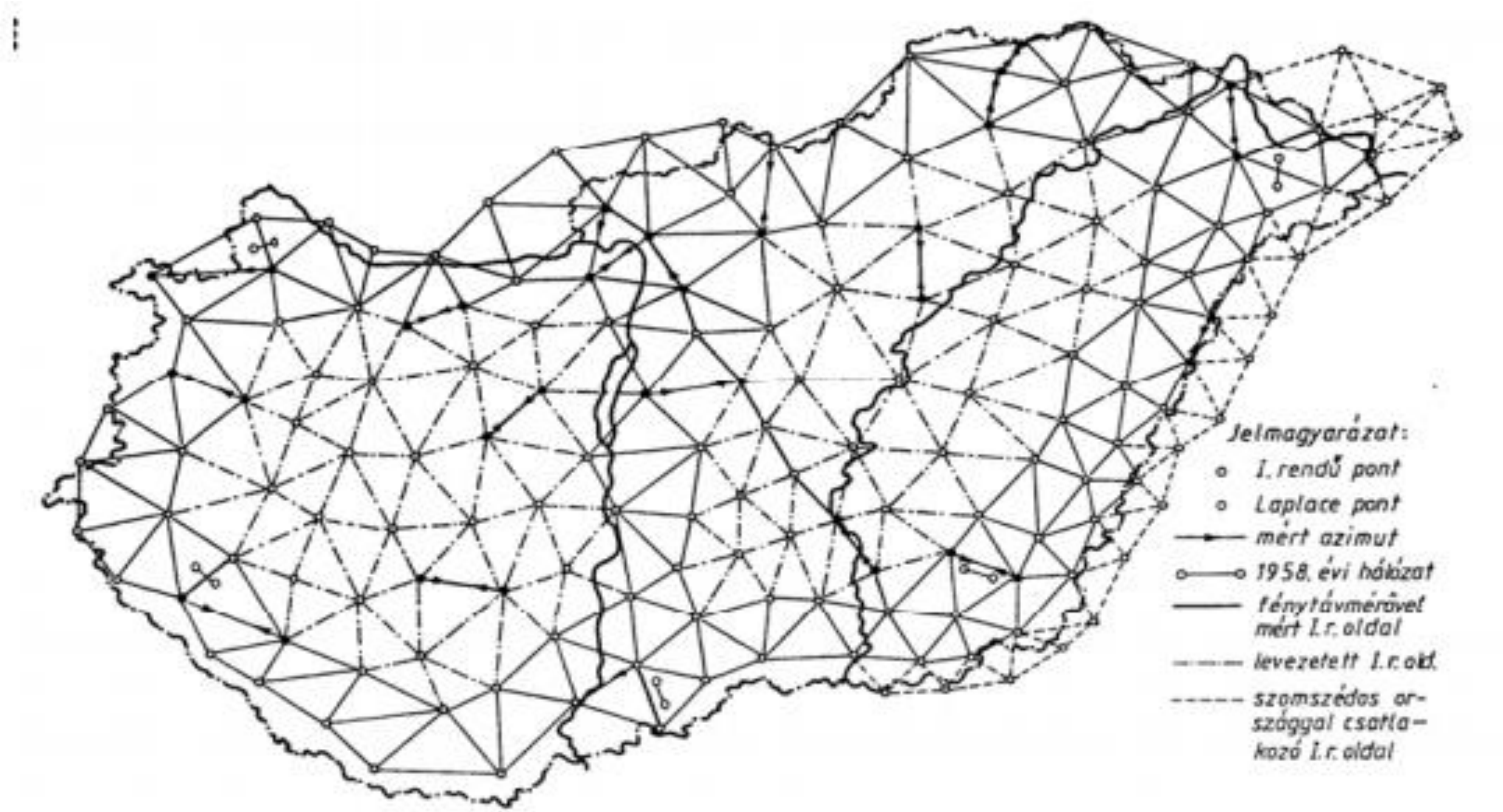
- In the environment of the applied Laplace-points, the surface of the geoid and this ellipsoid are approximately parallel: this is a *locally fitted ellipsoid; (*e.g., European Datum 1950, Pulkovo 1995 in Russia, NAD83 in the USA, SAD69 in Brazil, ED50 Egypt, Albanian 1987, HD72 in Hungary,…);



- The satellite geodesy enables to establish *globally* fitted ellipsoid, too (e.g., GRS80, WGS84 datum).



The astrogeodetic network of Hungary with the 37 Laplace points



Control point 1 (Lágymányos Campus Northern Building sun physics terrace centre point)

Geographic coordinates:

	Φ	**Λ**
WGS-84	**47° 28' 29.262"**	**19° 3' 43.303"**
GK-S42 ***(Krasovsky)***	**47° 28' 30.496"**	**19° 3' 49.135"**
HD72 ***(IUGG67)***	**47° 28' 30.206"**	**19° 3' 47.2824"**
HD1909 ***(Bessel)***	**47° 28' 26.076"**	**19° 3' 49.3709"**

Height above ellipsoid=187.575 m

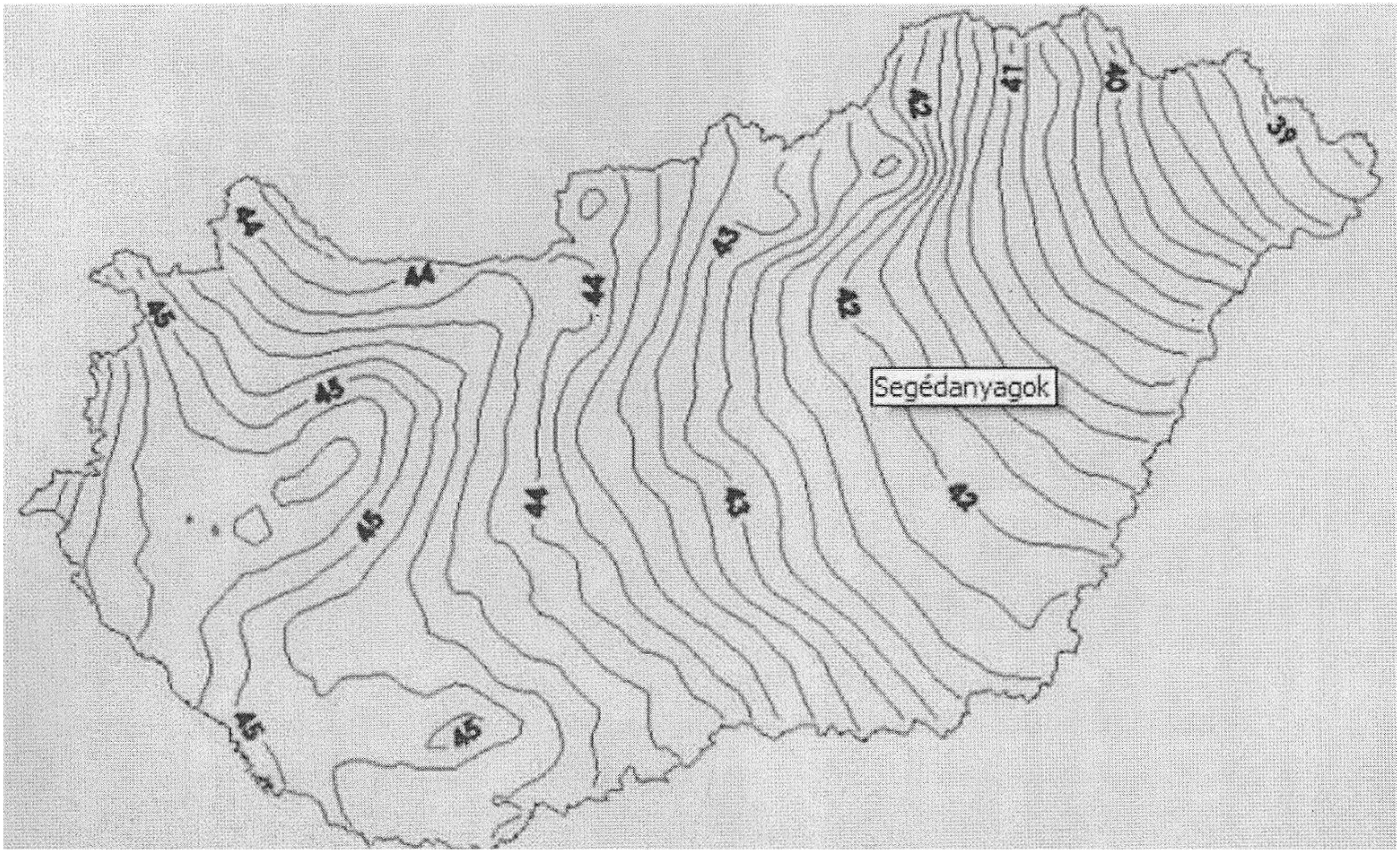
Map coordinates:

	Y	**X**
EOV	**651097.685 m**	**236759.995 m**
(UTM- *WGS84)*	(353975.846 m)	(5259748.201)
Bp. sztereo	**-1097.61 m**	**1345.08 m**

Height above geoid=143.755 m

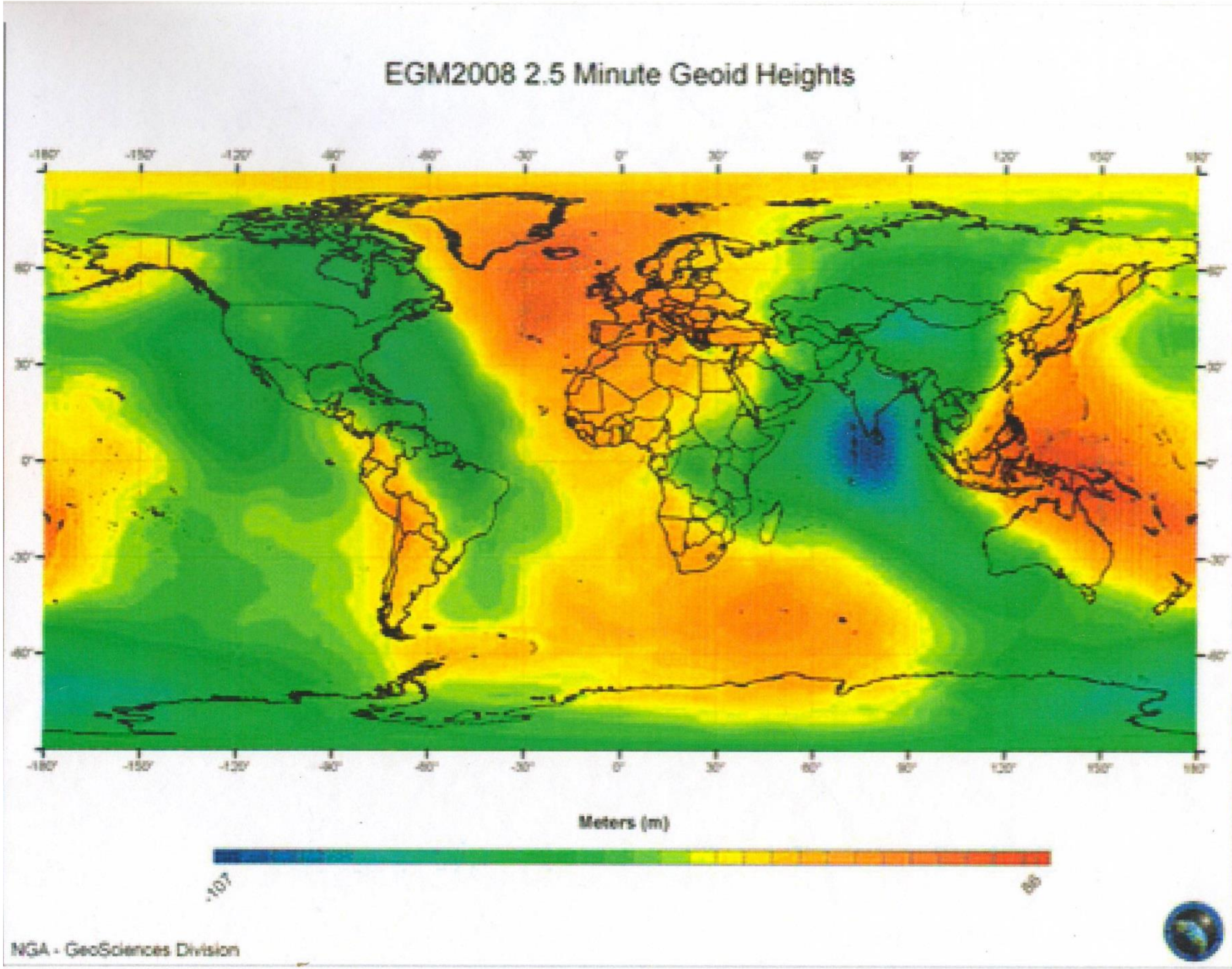
Undulation of the geoid

The difference $h_{ell}–h_{geoid}$ of the ellipsoidal height and the „normal height" (height above sea level) measured on the geoid is the „*undulation of the geoid*". Its value is between –106 m and 85 m in the case of WGS84 datum.



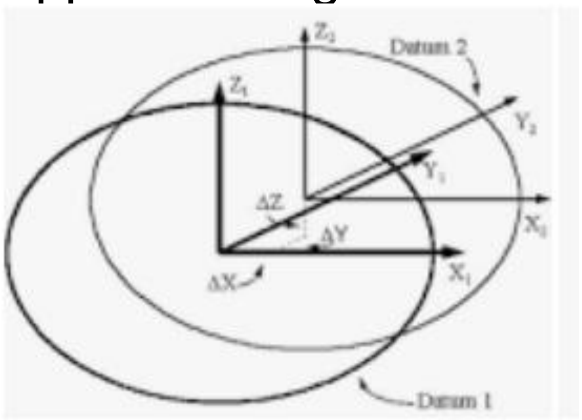
Contour map for the geoid undulation in Hungary

Contour map for the geoid undulation on earth



Transformations of coordinates between different reference surfaces (1)

- The Earth coordinates created in different times and in several circumstances apply to *different geodetic datums.* By their integration into a common reference system, some of them have to be transformed from a geodesic datum onto another one. Exact transformation between different geodetic datums doesn't exist in general, therefore approximating transformation methods were developed.
- **Three-parameter transformation directly** between **geographic coordinates** of different datums (**Molodenski** abridged transformation) – by **shifting** one datum to another. The spatial rectangular coordinates of the shift are: ΔX,ΔY,ΔZ



$$arc\Delta\Phi = \frac{-\Delta X \cdot \sin\Phi \cdot \cos\Lambda - \Delta Y \cdot \sin\Phi \cdot \sin\Lambda + \Delta Z \cdot \cos\Phi + (a \cdot \Delta f + f \cdot \Delta a) \cdot \sin(2\Phi)}{M(\Phi)}$$

$$arc\Delta\Lambda = \frac{-\Delta X \cdot \sin\Lambda + \Delta Y \cdot \cos\Lambda}{N(\Phi) \cdot \cos\Phi}$$

$$\Delta h = \Delta X \cdot \cos\Phi \cdot \cos\Lambda + \Delta Y \cdot \cos\Phi \cdot \sin\Lambda + \Delta Z \cdot \sin\Phi + (a \cdot \Delta f + f \cdot \Delta a) \cdot \sin^2\Phi - \Delta a$$

Transformations of coordinates between different reference surfaces (2)

- **Seven-parameter transformation** between **spatial rectangular coordinates** of different datums (**Burša-Wolf** transformation)

  It is composed of a *translation*, three *rotations* and a *similarity transformation* (scaling up or down). The seven parameters are: the components ΔX, ΔY, ΔZ of the *translation vector*, the *rotation angles* ε_x, ε_y and ε_z about the axes, and the *ratio* $(1+\kappa)$ of the *similarity transformation*. Supposing that the rotation angles are small, the formula below assigns the transformed coordinates X', Y', Z' of the spatial point given by coordinates X, Y, Z:

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} + (1+\kappa) \cdot \begin{bmatrix} 1 & \varepsilon_z & -\varepsilon_y \\ -\varepsilon_z & 1 & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

- Both transformations demand *fitting points* with coordinates known in both reference system. The parameters of the transformations can be obtained from the fitting points.

Substituting the ellipsoid by sphere

Under the map scale cca. 1:1 million (mainly among geographic maps), the coordinate differences on the map caused by the eccentricity of the ellipsoid are smaller than the maximum accuracy of the topographic map (0.1–0.2 mm). So the geocartography uses rather spherical coordinates instead of ellipsoidal, in order to simplify the projection calculations.

The radius R of the sphere can be calculated e.g. by the average of values of meridian radius of curvature which varies depending on the latitudes of points on the area of representation:

$$R = \frac{\int_{\emptyset_S}^{\emptyset_N} M(\emptyset)\, d\emptyset}{arc(\emptyset_N - \emptyset_S)} = \frac{\int_{\emptyset_S}^{\emptyset_N} \frac{a \cdot (1 - e^2)}{(1 - e^2 \cdot sin^2\emptyset)^{3/2}}\, d\emptyset}{arc(\emptyset_N - \emptyset_S)}$$

where Φ_S and Φ_N are the bounding latitudes. The radius concerning the medium latitude Φ_m of the area to be represented is in use, too:

$$R = \sqrt{M(\emptyset_m) \cdot N(\emptyset_m)}$$

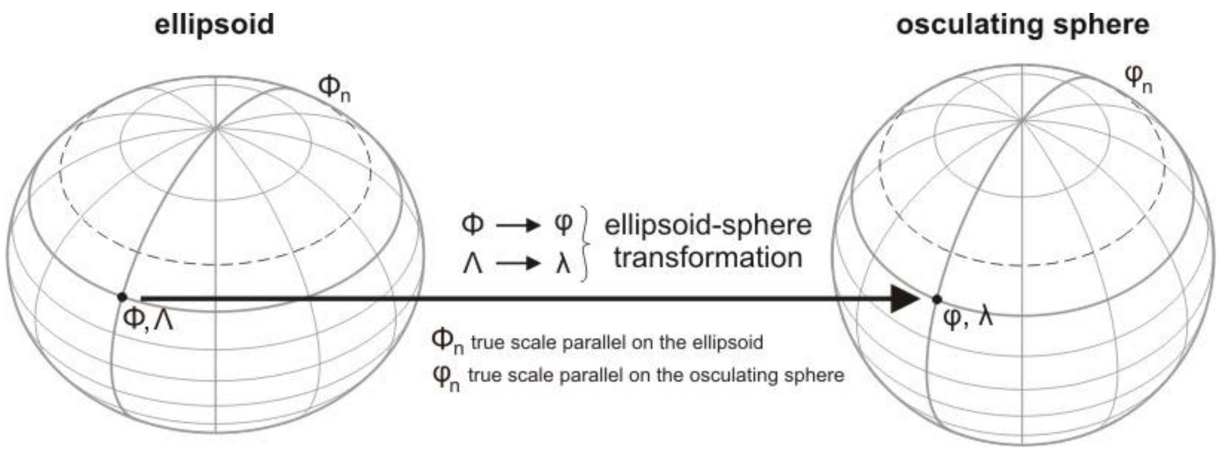
An other usual simple approximation of the mean value of the radius R:

$$R = \frac{2 \cdot a + b}{3}$$

The transformation of an ellipsoidal surface onto a sphere (ellipsoid-sphere transformation)

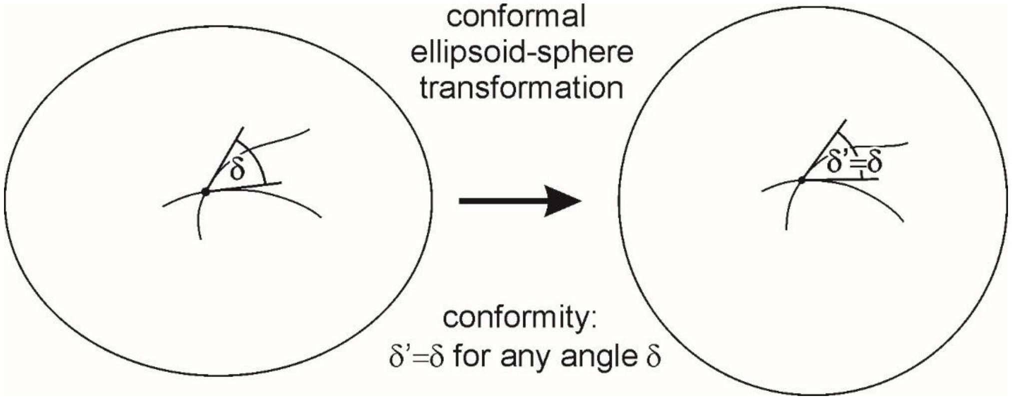
Some map projections from an ellipsoidal surface onto the plane use an intermediate sphere. In this case a special *ellipsoid-sphere transformation* assigns the points of a locally approximating sphere („*osculating sphere*") to the points of an ellipsoid under standard differentiable conditions. Requirements on its basic properties:

- The images of the ellipsoidal parallels are spherical parallels, in other words the spherical latitude depends only the ellipsoidal latitude: $\varphi = \varphi(\Phi)$.
- The images of the ellipsoidal meridians are spherical meridians, and the longitude differences on the ellipsoid are proportional to the correspondent longitude differences on the sphere, in other words: $\Delta\lambda = n \cdot \Delta\Lambda$, where *n* is constant.



Conformal mapping from ellipsoid onto sphere (1)

Such most commonly used mappings can be *conformal* (the angles do not change locally during the mapping), *equal-area* (the area of the objects on the ellipsoidal surface are equal to the area of the correspondent objects on the spherical surface), or *the linear scale is true along the meridians* (the lengths of the meridian arcs on the ellipsoid are equal to the lengths of the correspondent meridian arcs on the sphere).



The geodesy and geoinformatics (GIS) use mainly conformal mapping for the ellipsoid-sphere transformation. The conformity (angle invariance) means that *any angle of the tangents of intersecting smooth ellipsoidal surfacial curves in the intersection will be the same as the angle of the tangents of the corresponding spherical surfacial curves in the intersection*.

Conformal mapping from ellipsoid onto sphere (2)

The equations originating in this condition result in the connection below between the differing spherical latitude φ and ellipsoidal latitude Φ:

$$\tan\left(45° + \frac{\varphi}{2}\right) = \kappa \cdot \tan^n\left(45° + \frac{\Phi}{2}\right) \cdot \left(\frac{1 - e \cdot \sin\Phi}{1 + e \cdot \sin\Phi}\right)^{\frac{n \cdot e}{2}}$$

Based on this equation, the spherical latitude φ can be calculated from the ellipsoidal latitude Φ by the formula below:

$$\varphi = 2 \cdot \arctan\left[\kappa \cdot \tan^n\left(45° + \frac{\Phi}{2}\right) \cdot \left(\frac{1 - e \cdot \sin\Phi}{1 + e \cdot \sin\Phi}\right)^{\frac{n \cdot e}{2}}\right] - 90°$$

Reversaly, there is an iteration formula for the ellipsoidal latitude from the spherical latitude with the expected accuracy ε and initial value of $\Phi'' = \varphi$. Φ' gets the value of Φ'', then

$$\Phi'' = 2 \cdot \arctan \sqrt[n]{\frac{\tan\left(45° + \frac{\varphi}{2}\right)}{\kappa \cdot \left(\frac{1 - e \cdot \sin\Phi'}{1 + e \cdot \sin\Phi'}\right)^{\frac{e \cdot n}{2}}}} - 90°$$

These are repeated until $|\Phi' - \Phi''| > \varepsilon$. Finally, $\Phi = \Phi''$ is the ellipsoidal latitude.

Ellipsoidal longitude as usual: $\Lambda = \lambda / n$

Conformal mapping from ellipsoid onto sphere (3)

The constants κ, n and the radius R depend on the prescribed *true-scale parallel* (generally the medium latitude of the area to be represented). Let Φ_S be its latitude on the ellipsoid and φ_S on the sphere, then the *linear scale* l_p (proportion between the correspondent spherical parallel arc length Δp' and ellipsoidal parallel arc length Δp) along this parallel is:

$$l_p(\Phi_s, \varphi_s) = \frac{\Delta p'}{\Delta p} = \frac{R \cdot \cos\varphi_s \cdot \Delta\lambda}{N(\Phi_s) \cdot \cos\Phi_s \cdot \Delta\Lambda} = \frac{R \cdot \cos\varphi_s \cdot n \cdot \Delta\Lambda}{N(\Phi_s) \cdot \cos\Phi_s \cdot \Delta\Lambda} = \frac{R \cdot \cos\varphi_s \cdot n}{N(\Phi_s) \cdot \cos\Phi_s}$$

The basic requirement is the *existence* of a true scale parallel (l_p=1). In addition the *scale distortion* (the change in lengths due to the mapping) on the represented territory about the true scale parallel should be minimal. The condition was formulated and the problem solved by the matematician Gauss. The resulting system of equations for the variables Φ_S, φ_S,n and R:

$$n \cdot \sin\varphi_s = \sin\Phi_s$$

$$\sqrt{1+(e')^2 \cdot \cos^2\Phi_s} \cdot \tan\varphi_s = \tan\Phi_s$$

$$R = \sqrt{M(\Phi_s) \cdot N(\Phi_s)}$$

After prescribing Φ_S or φ_S , the other variables can be calculated from this system of equation. Finally,

$$\kappa = \frac{\tan\left(45° + \frac{\varphi_s}{2}\right)}{\tan^n\left(45° + \frac{\Phi_s}{2}\right) \cdot \left(\frac{1 - e \cdot \sin\Phi_s}{1 + e \cdot \sin\Phi_s}\right)^{\frac{e \cdot n}{2}}}$$

Home work

1. Given: the metapole N* (φ_0=47.5°, λ_0=19°) and the geographic coordinates (φ_P=41° 0'18.60", λ_P=28°58'28.79") of the point P (Ayasofiya, Istanbul).

   Asked: the metacoordinates φ^*, λ^*.

2. Given: coordinates of the intersection point K of the metaequator and the prime metameridian (φ_K=47.5°, λ_K=19°) and the metacoordinates of the point P (Rome) ($\varphi^*=-5.40287°$, $\lambda^*=-4.8668°$).

   Asked: the geographic coordinates of the point P (φ_P,λ_P).

3. Mount Ida (Island Crete, Greece) GPS coordinates:

   φ=35°13'23.07" N; λ=24°46'9.12" E; normal height: 2456m

   What is the elevation above the ellipsoid?

   (e.g., https://earth-info.nga.mil/index.php?dir=wgs84&action=egm96-geoid-calc Run)

4. Which are the geographic coordinates of the Mount Ida on the S42 (Krasovskiy) datum, if $\Delta X=-28$m, $\Delta Y=130$m, $\Delta Z=95$m?

   What is the undulation related to S42?