Lesson_6

- Supplement: conformal mapping the ellipsoidal surface onto an intermediate sphere
- Map projections and map distortions in general

Supplement to the ellipsoid-sphere transformations

- The *Tissot theory* of the map projections investigates mappings from the reference system to the plane under appropriate differentiability conditions, but it can be related also to the ellipsoid-sphere transformation.

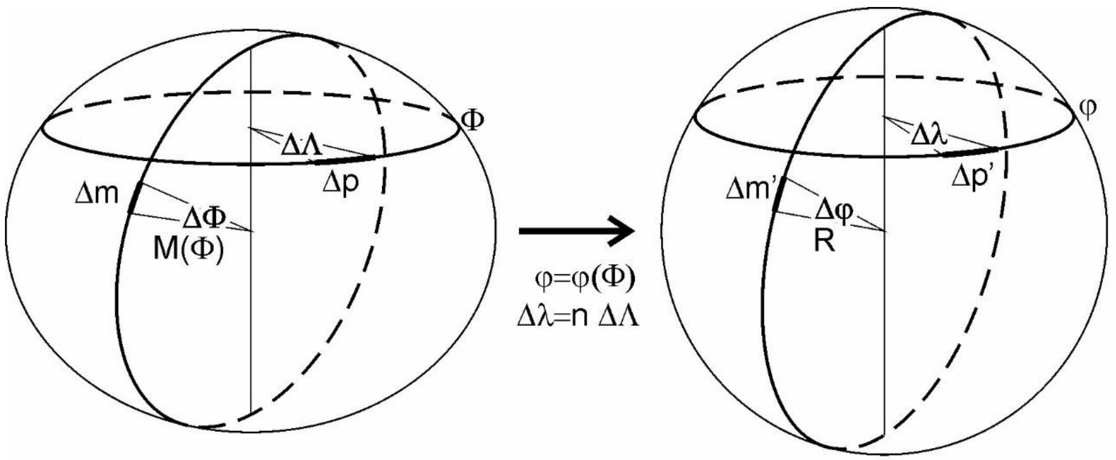
- Definition: the *principal directions* are correspondent directions on the initial reference surface (e.g. ellipsoidal surface) and on the target surface (e.g. spherical surface) which are *perpendicular* on both surfaces.

- Theorem: the linear scale (change in lengths) is *maximal* along one of these directions, and *minimal* along the other direction.

- The directions of the *graticule lines* are perpendicular both on the ellipsoid and on the sphere, so they provide the principal directions.

- If the maximum and minimum linear scale are equal at a definite point of the mapping then the changes of the lengths are the same in every direction. This property is equivalent to the *local angle invariance* (conformity) at this point.

The linear scales along the graticule lines in ellipsoid-sphere transformation



$$h = \lim_{\Delta\Lambda \to 0} \frac{\Delta p'}{\Delta p} = \lim_{\Delta\Lambda \to 0} \frac{R \cdot \cos\varphi \cdot arc\Delta\lambda}{N(\Phi) \cdot \cos\Phi \cdot arc\Delta\Lambda} = \frac{R \cdot \cos\varphi \cdot n}{N(\Phi) \cdot \cos\Phi}$$

$$k = \lim_{\Delta\Phi \to 0} \frac{\Delta m'}{\Delta m} = \lim_{\Delta\Phi \to 0} \frac{R \cdot arc\Delta\varphi}{M(\Phi) \cdot arc\Delta\Phi} = \frac{R}{M(\Phi)} \cdot \frac{d\varphi}{d\Phi}$$

Conformal mapping from ellipsoid onto sphere (1)

- The equation of the conformity based on the foregoing: *h = k* ,

that is

$$\frac{R}{M(\Phi)} \cdot \frac{d\varphi}{d\Phi} = \frac{R \cdot \cos\varphi \cdot n}{N(\Phi) \cdot \cos\Phi}$$

- Rearranging: $$\frac{d\varphi}{d\Phi} = \frac{M(\Phi) \cdot \cos\varphi \cdot n}{N(\Phi) \cdot \cos\Phi} = n \cdot \frac{a \cdot (1 - e^2) \cdot \cos\varphi \cdot \sqrt{1. - e^2 \cdot \sin^2\Phi}}{(1. - e^2 \cdot \sin^2\Phi)^{\frac{3}{2}} \cdot a \cdot \cos\Phi}$$

- It is a separable differential equation: $$\frac{d\varphi}{d\Phi} = n \cdot \frac{(1 - e^2) \cdot \cos\varphi}{(1. - e^2 \cdot \sin^2\Phi) \cdot \cos\Phi}$$

- Integration: $$\int \frac{d\varphi}{\cos\varphi} = n \cdot \int \frac{(1 - e^2)}{\cos\Phi \cdot (1. - e^2 \cdot \sin^2\Phi)} \, d\Phi$$

  that is

$$\ln\tan\left(45^\circ + \frac{\varphi}{2}\right) = n \cdot \int \frac{(1 - e^2 \cdot \sin^2\Phi - e^2 \cdot \cos^2\Phi)}{\cos\Phi \cdot (1. - e^2 \cdot \sin^2\Phi)} \, d\Phi = n \cdot \int \frac{1 - e^2 \cdot \sin^2\Phi}{\cos\Phi \cdot (1. - e^2 \cdot \sin^2\Phi)} \, d\Phi + n \cdot \int \frac{-e^2 \cdot \cos^2\Phi}{\cos\Phi \cdot (1. - e^2 \cdot \sin^2\Phi)} \, d\Phi$$

Conformal mapping from ellipsoid onto sphere (2)

- Solving the equation:

$$\ln\tan\left(45^\circ+\frac{\varphi}{2}\right)=n\cdot\int\frac{1}{\cos\Phi}\,d\Phi-n\cdot\int\frac{e^2\cdot\cos\Phi}{\left(1.-e^2\cdot\sin^2\Phi\right)}\,d\Phi=n\cdot\ln\tan\left(45^\circ+\frac{\Phi}{2}\right)+n\cdot\frac{e}{2}\cdot\ln\left(\frac{1-e\cdot\sin\Phi}{1+e\cdot\sin\Phi}\right)+\ln\kappa$$

because $\dfrac{d\left[\dfrac{e}{2}\cdot\ln\left(\dfrac{1-e\cdot\sin\Phi}{1+e\cdot\sin\Phi}\right)\right]}{d\Phi}=-\dfrac{e^2\cdot\cos\Phi}{1-e^2\cdot\sin^2\Phi}$ (extra credit example to check it)

and ln*k* is the constant of integration.

- Applying the function exp (the inverse function of the ln) for both side of the equation:

$$\tan\left(45^\circ+\frac{\varphi}{2}\right)=\kappa\cdot\tan^n\left(45^\circ+\frac{\Phi}{2}\right)\cdot\left(\frac{1-e\cdot\sin\Phi}{1+e\cdot\sin\Phi}\right)^{\frac{n\cdot e}{2}}$$

- which provides the relationship between the ellipsoidal latitude Φ and the spherical latitude φ for the conformal ellipsoid-sphere transformation, and as usual,

$$\lambda=n\cdot\Lambda$$

Conformal mapping from ellipsoid onto sphere (3)

Practically: given Φ, asked φ: from the equation above φ can be expressed:

$$\varphi = 2 \cdot \arctan\left[\kappa \cdot \tan^n\left(45° + \frac{\Phi}{2}\right) \cdot \left(\frac{1 - e \cdot \sin\Phi}{1 + e \cdot \sin\Phi}\right)^{\frac{n \cdot e}{2}} \right] - 90°$$

Reversaly, given φ, asked Φ: Φ from the factor below can be expressed

$$\tan^n\left(45° + \frac{\Phi}{2}\right)$$

which is in the product of the right side of the equation above. (It is the *succesive substitution* method for solving nonlinear equations).

$$\Phi'' = 2 \cdot \arctan \sqrt[n]{\frac{\tan\left(45° + \frac{\varphi}{2}\right)}{\kappa \cdot \left(\frac{1 - e \cdot \sin\Phi'}{1 + e \cdot \sin\Phi'}\right)^{\frac{e \cdot n}{2}}}} - 90°$$

Let the expected accuracy be ε; chosen the initial value for Φ'' e.g. as $\Phi''=\varphi$. Φ' gets the value of Φ'', then the previous formula provides a better Φ''-.

These are repeated until $|\Phi' - \Phi''| > \varepsilon$. Finally, $\Phi = \Phi''$ is the ellipsoidal latitude.

Example Ellipsoid-sphere transformation constants: κ=1.0031100083, n=1.0007197049, e=0.0818205679. *Given*: Bp point coordinates (HD72 datum, IUGG67 ellipsoid): Φ=47°28’30.206”, Λ=0°0’52.4241” (+19°2’54.8584”). *Asked*: spherical coordinates φ and λ.

Map projections in general (1)

Map coordinates based on map projections

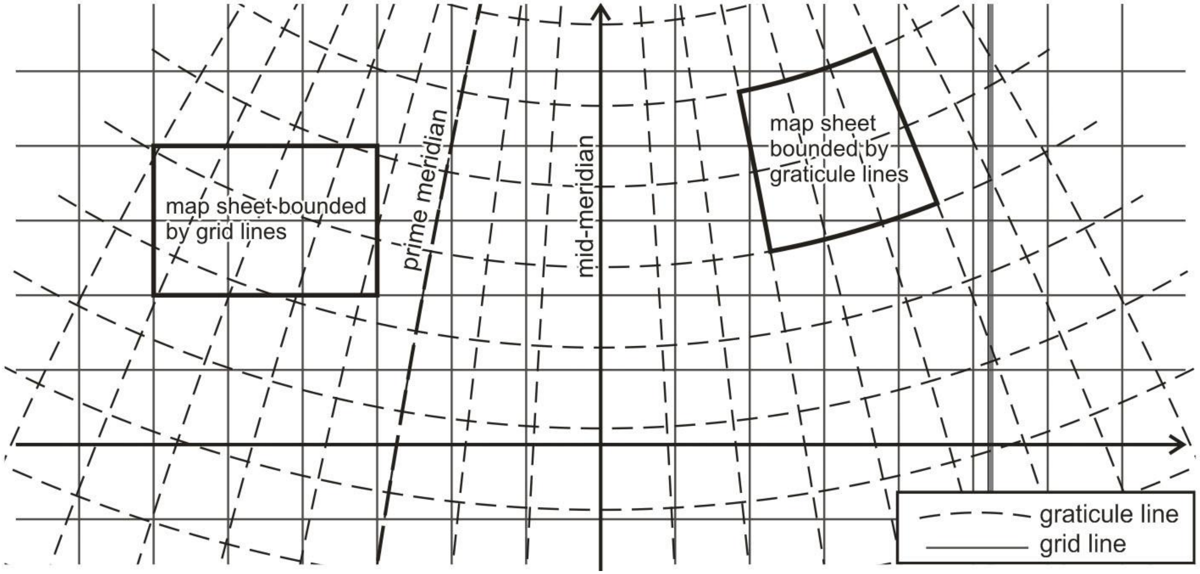
In addition to orientation, the large scale maps are used to locate of point of earth surface objects and to determine their dimensions.

The *location* of military or civil objects requires accurate coordinates. The images of graticule lines on large scale maps are mostly composed of imperceptibly curved lines, so they do not offer an opportunity to determine exactly the geographic coordinates. Therefore the large scale maps in most of the cases have a *planar rectangular map coordinate system*. The *map projection* gives the connection between the earth and map coordinates mathematically, these are the *projection equations*. With the help of measured rectangular map coordinates of a point, its real Earth location can be deduced back by the projection equations which assigns map coordinates to the earth coordinates and reversaly.

Sometimes a *planar polar coordinate system* can be created on a map, mostly as an intermediate system.

Map projections in general (2)

Graticule and grid lines



The graticule lines and the rectangular coordinate lines (*grid* lines) on the map are generally not parallel to each other. The map graticule has at least one appointed axis of reflection symmetry coinciding with a meridian, the so called „mid-meridian" or „central meridian" (mostly not identical with the prime meridian), which runs along the midline of the area to be represented. The direction of this axis is the „grid north", while the tangent of the meridian gives the „true north" (in other words „geodetic north") at every point of the map.

In most instances the topographic map sheets are bounded by graticule lines, while some kinds of cadastral maps are bounded by grid lines.

Map projections in general (3)

- **Transformation of coordinates from the Earth to the map**

The mathematical connection between the earth and map coordinates is realized by the *projection equations* (they are actually functions), which assign the planar map coordinates *x,y* to the geographic coordinates of an earth point:

$x = x(\varphi, \lambda)$ and $y = y(\varphi, \lambda)$ in case of spherical reference surface, or
$x = x(\Phi, \Lambda)$ and $y = y(\Phi, \Lambda)$ in case of ellipsoidal reference surface.

If the projection equations, defined on spherical coordinates do not refer to geographical but to metacoordinates:

$x = x(\varphi^*, \lambda^*)$ and $y = y(\varphi^*, \lambda^*)$

there is a *transverse* or in other words *equatorial* (metapole is on the equator) or *oblique* (metapole is neither on the equator nor on the original pole) projection - as opposed to the *normal* projection above.

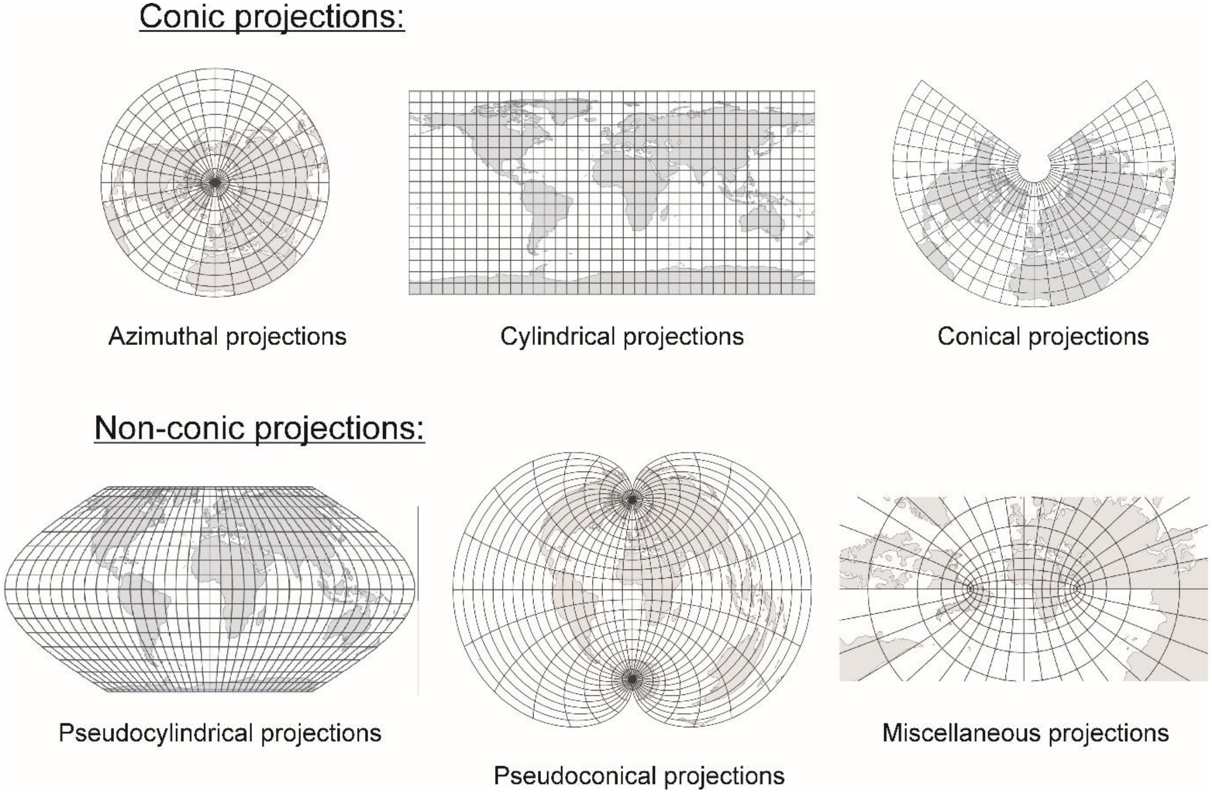
The *invers projection equations* assign the geographic coordinates to the planar map coordinates: $\varphi = \varphi(x, y)$ and $\lambda = \lambda(x, y)$ in spherical case, or
$\Phi = \Phi(x, y)$ and $\Lambda = \Lambda(x, y)$ in ellipsoidal case.

Requirements for the direct and inverse projection equations are: *injectivity*, *twice continously differentiability* and *describability by formulae or series*.

Map projections in general (4)

The main classes of the map projections

The classification of the map projections is founded on the geometric character of the graticule (or metagraticule) on the map. *Conic projections*: meridians as straight lines, and parallels as circles, parallel straight lines or circular arcs. If any of these properties are missing: *non-conic projections.* The large scale maps (so the maps of the GIS) use mainly conic projections.

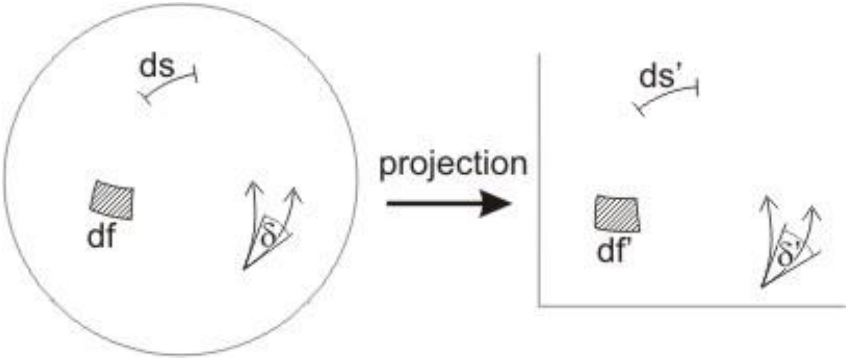


Map distortions (1)

One of the main purposes of map use is to *infer* the size of earth surface *objects* (arcs, figures, directions) by measuring the size of corresponding map objects. In the course of the planar representation of these objects, some of their *sizes* (lengths, areas, directions, angles) *change*. This makes the measuring the shape sizes and earth coordinates difficult in practice, and is mathematically formulated in the so called *local map distortions*, viz. the linear scale *l*, area scale *p* and angular distortion *i*, one after the other:

$$l = \frac{ds'}{ds} \qquad p = \frac{df'}{df} \qquad i = \frac{\tan\delta'}{\tan\delta}$$

where ds, ds', df, df' are infinitesimal quantities, in addition δ and δ' are the corresponding angles on the reference surface and on the map plane, respectively.



The Earth cannot be represented on a map without scale distortions, but there are projections without area distortions (*equal-area* or *equivalent* projections), or without angular distortions (*conformal* projections).

Home work:

Ellipsoid-sphere transformation constants: κ=1.0031100083,

n=1.0007197049, e=0.0818205679

Given the spherical coordinates of a point :

φ=47°25'48.5686", λ=0°0'52.4618" (+19°2'54.8584").

Asked ellipsoidal coordinates Φ and Λ (HD72 datum, IUGG67 ellipsoid).

Required accuracy: ε=0.1m, or at least 3 iteration steps.