Lesson_7

- Calculation of local map distortions
- Azimuthal map coordinate systems
- The stereographic projection of the sphere
- Conformal azimuthal („stereographic”) projection of the ellipsoid
- Double stereographic projection

Map distortions (2)

Based on the *projection equations* of a map projection, the local distortions at each point on the map can be determined in a few steps.

1. calculation of the *graticule distortions*, namely the linear scale *h* along the parallel, the linear scale *k* along the meridian, and the angle Θ between the images of the graticule lines on the appointed location (see next page):

$$h = \frac{\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2}}{R \cdot \cos\varphi} \qquad k = \frac{\sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2}}{R} \qquad \cot\Theta = \frac{\frac{\partial x}{\partial \lambda}\cdot\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial \lambda}\cdot\frac{\partial y}{\partial \varphi}}{\frac{\partial x}{\partial \lambda}\cdot\frac{\partial y}{\partial \varphi} - \frac{\partial y}{\partial \lambda}\cdot\frac{\partial x}{\partial \varphi}}$$

and

$$\sin\Theta = \frac{1}{\sqrt{1+\cot^2\Theta}}$$

The scale distortions along the graticule lines in case of ellipsoidal projection:

$$h = \frac{\sqrt{\left(\frac{\partial x}{\partial \Lambda}\right)^2 + \left(\frac{\partial y}{\partial \Lambda}\right)^2}}{N(\Phi)\cdot\cos\Phi} \qquad k = \frac{\sqrt{\left(\frac{\partial x}{\partial \Phi}\right)^2 + \left(\frac{\partial y}{\partial \Phi}\right)^2}}{M(\Phi)}$$

2. calculation of maximum and minimum linear scales, denoted by *a* and *b*, being perpendicular to each other, both on the reference surface and on the map which is done using graticule distortions:

$$a = \frac{\sqrt{h^2 + k^2 + 2\cdot h\cdot k\cdot\sin\Theta} + \sqrt{h^2 + k^2 - 2\cdot h\cdot k\cdot\sin\Theta}}{2}$$

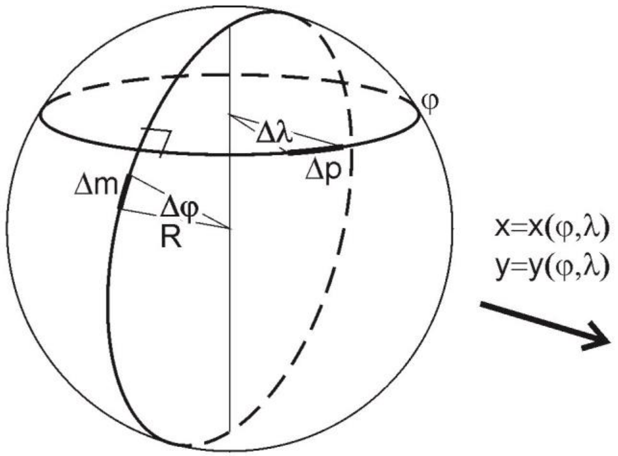
and

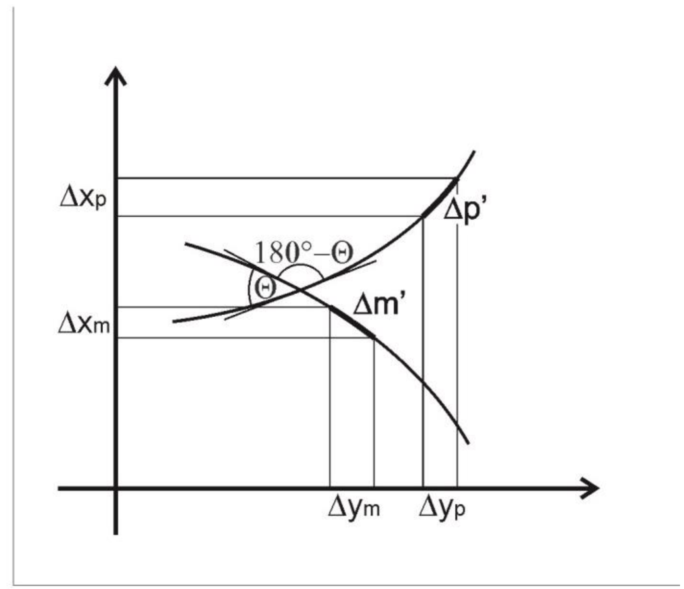
$$b = \frac{\sqrt{h^2 + k^2 + 2\cdot h\cdot k\cdot\sin\Theta} - \sqrt{h^2 + k^2 - 2\cdot h\cdot k\cdot\sin\Theta}}{2}$$

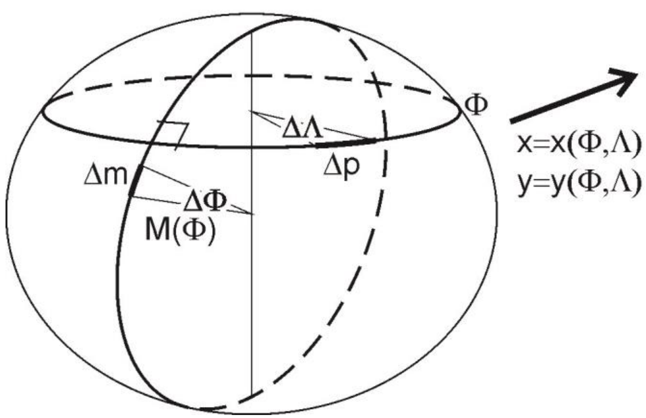
The linear scales along the graticule lines in map projections

$$h=\lim_{\Delta\lambda\to 0}\frac{\Delta p'}{\Delta p}=\lim_{\Delta\lambda\to 0}\frac{\sqrt{(\Delta x_p)^2+(\Delta y_p)^2}}{R\cdot\cos\varphi\cdot arc\Delta\lambda}=\lim_{\Delta\lambda\to 0}\frac{\sqrt{\left(\frac{\Delta x_p}{arc\Delta\lambda}\right)^2+\left(\frac{\Delta y_p}{arc\Delta\lambda}\right)^2}\cdot arc\Delta\lambda}{R\cdot\cos\varphi\cdot arc\Delta\lambda}=\frac{\sqrt{\left(\frac{\partial x}{\partial\lambda}\right)^2+\left(\frac{\partial y}{\partial\lambda}\right)^2}}{R\cdot\cos\varphi}$$

$$k=\lim_{\Delta\varphi\to 0}\frac{\Delta m'}{\Delta m}=\lim_{\Delta\varphi\to 0}\frac{\sqrt{(\Delta x_m)^2+(\Delta y_m)^2}}{R\cdot arc\Delta\varphi}=\lim_{\Delta\varphi\to 0}\frac{\sqrt{\left(\frac{\Delta x_m}{arc\Delta\varphi}\right)^2+\left(\frac{\Delta y_m}{arc\Delta\varphi}\right)^2}\cdot arc\Delta\varphi}{R\cdot arc\Delta\varphi}=\frac{\sqrt{\left(\frac{\partial x}{\partial\varphi}\right)^2+\left(\frac{\partial y}{\partial\varphi}\right)^2}}{R}$$







$$h=\lim_{\Delta\Lambda\to 0}\frac{\Delta p'}{\Delta p}=\lim_{\Delta\Lambda\to 0}\frac{\sqrt{(\Delta x_p)^2+(\Delta y_p)^2}}{N(\Phi)\cdot\cos\Phi\cdot arc\Delta\Lambda}=\lim_{\Delta\Lambda\to 0}\frac{\sqrt{\left(\frac{\Delta x_p}{arc\Delta\Lambda}\right)^2+\left(\frac{\Delta y_p}{arc\Delta\Lambda}\right)^2}\cdot arc\Delta\Lambda}{N(\Phi)\cdot\cos\Phi\cdot arc\Delta\Lambda}=\frac{\sqrt{\left(\frac{\partial x}{\partial\Lambda}\right)^2+\left(\frac{\partial y}{\partial\Lambda}\right)^2}}{N(\Phi)\cdot\cos\Phi}$$

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Map distortions (3)

3. calculation of the area scale, linear scale and angular distortion by *a, b.*

The formula for the area scale p is:

$$p = a \cdot b = h \cdot k \cdot \sin \Theta$$

Denoting the angle between the direction of the maximum scale distortion *a* and a questionable direction on the reference surface (first principal direction) by δ, the linear scale along the latter direction is:

$$l = \sqrt{a^2 \cdot \cos^2 \delta + b^2 \cdot \sin^2 \delta}$$

If one of the arms of an arbitrary angle coincides with the direction of the maximum linear scale *a*, then the angular distortion i refering to this angle: $i = \frac{b}{a}$

In consequence of the points described above, the *conformity* that is the equality of $\delta' = \delta$ for every angle δ at any point of the map, can be given by the equation

$a = b$ (equation of the conformity);

and similarly the *equivalency* at any point of the map by

$a \cdot b = 1$ (equation of the equivalency);

the *true scale* property at a definite point of the map in a definite direction whose angle closed by the first principal direction equals δ:

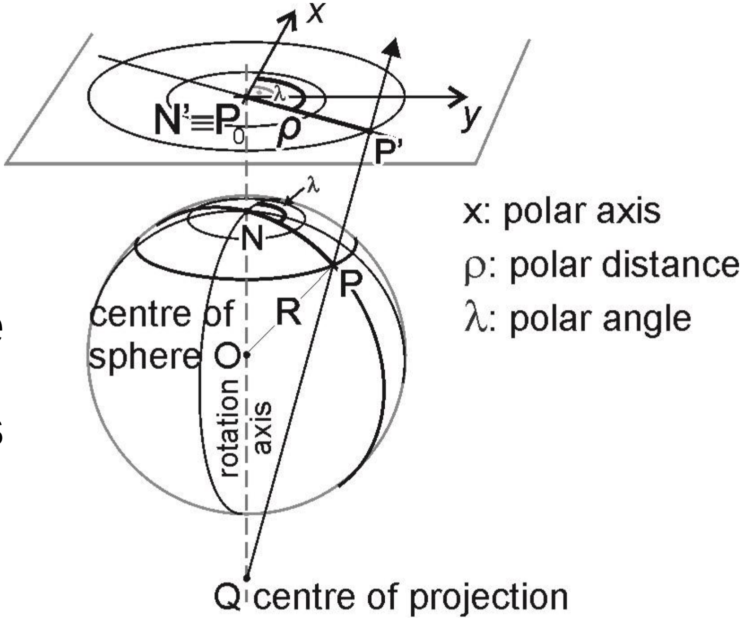
$$\sqrt{a^2 \cdot cos^2\delta + b^2 \cdot sin^2\delta} = 1$$

Perspective mapping of the sphere onto a plane

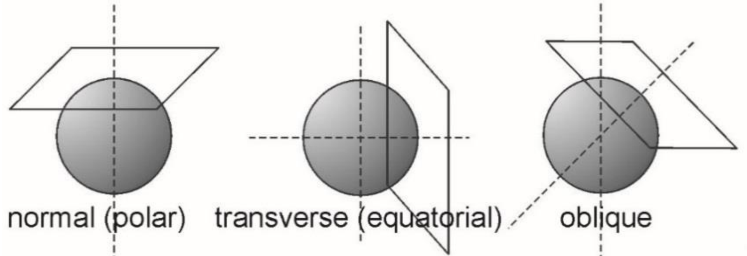
The points of the spherical surface are **projected** from a fixed point onto a plane by **straight lines** („projectors”). The prescribed reflection symmetry of the map graticule leads to the next properties:

- the centre of projection is located on the **rotation axis** of the earth sphere
- the plane is perpendicular to this rotation axis

Then the map graticule is composed of **concentric circles** (parallels) and **straight lines** (meridians) crossing the common centre of the circles, they are **perpendicular** to each other and the **angles between the meridians** are the same both on the sphere and the map plane – **perspective azimuthal (planar) map projections.**

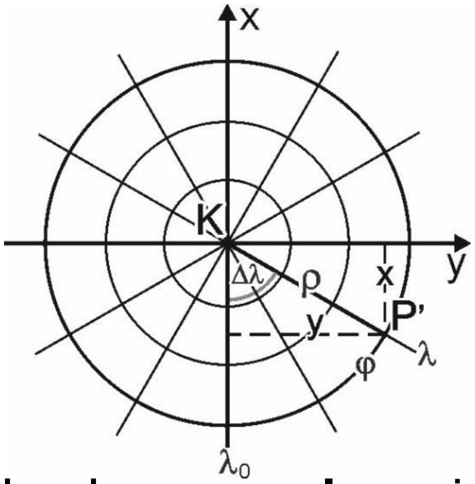


The system, composed of the centre and axis of projection as well as the map plane, can be rotated about the centre of the sphere. Versions depending on the mutual situation of the sphere and the map plane:

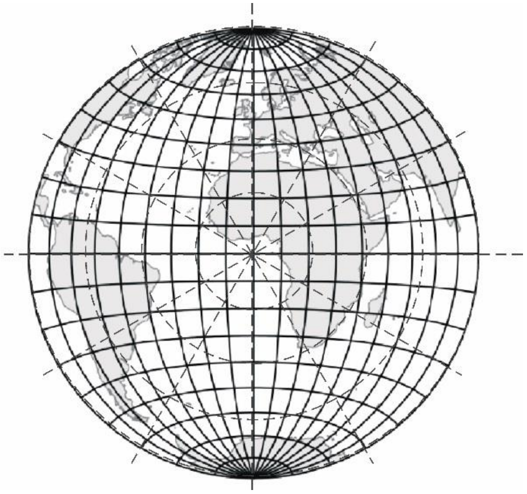


Azimuthal map projections of the sphere (1)

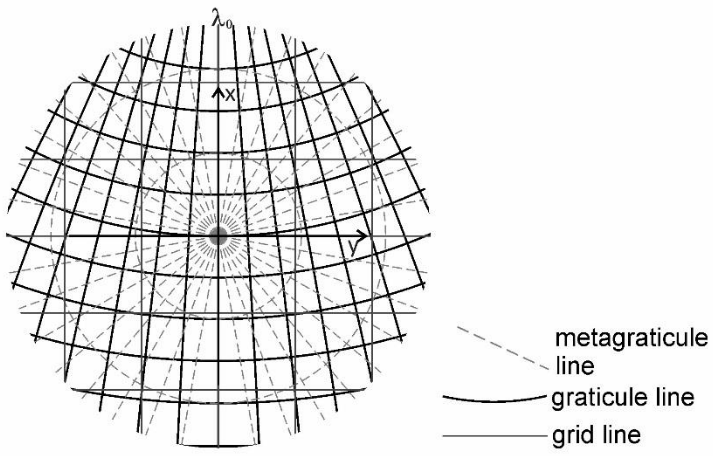
- If the map graticule (or metagraticule) has the previous properties then it is called **azimuthal map projection**;
- Polar coordinate system on the map: **origin** is in the common centre of the parallels (or metaparallels), **polar axis** coincides with the midmeridian λ_0 (or prime metameridian). **Polar distance** ρ is the radius of the parallel depending on the latitude φ (or metalatitude φ^*), **polar angle** is the longitude difference $\lambda - \lambda_0$ (or metalongitude λ^*).



Metapole coincides with the original pole: **normal** version.



metapole on the equator:
transverse version



metapole between the equator and the pole: **oblique version**

Azimuthal map projections of the sphere (2)

- The rectangular coordinates (giving the projection equations):

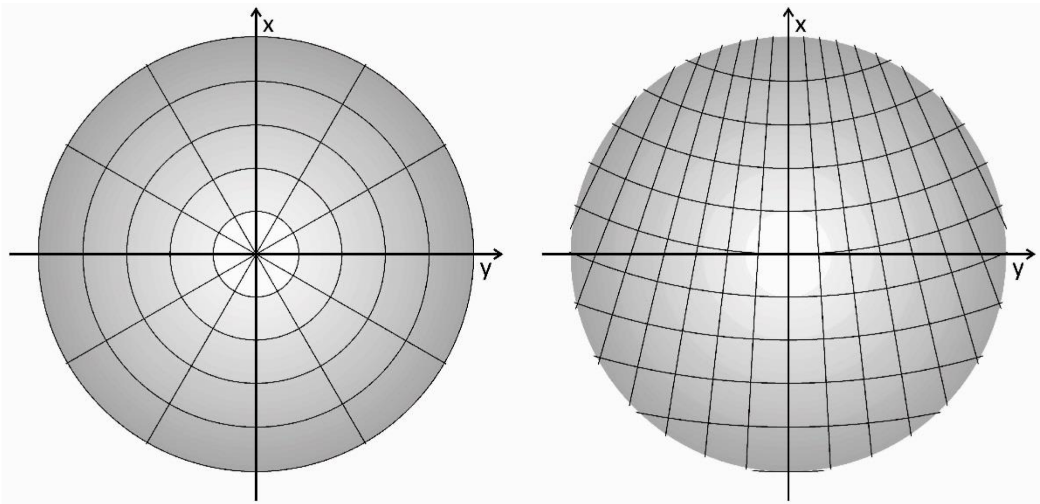
$x = \rho(\varphi) \cdot \cos(\lambda - \lambda_0)$ normal
$y = -\rho(\varphi) \cdot \sin(\lambda - \lambda_0)$ version

$x = \rho(\varphi^*) \cdot \cos(\lambda^*)$ oblique or
$y = -\rho(\varphi^*) \cdot \sin(\lambda^*)$ transverse version

- The graticule (or metagraticule) distortions are:

$$h = \frac{\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2}}{R \cdot \cos\varphi} = \frac{\rho(\varphi)}{R \cdot \cos\varphi} \qquad k = \frac{\sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2}}{R} = -\frac{d\rho}{d\varphi}\frac{1}{R}$$ and $\cot\Theta = 0$

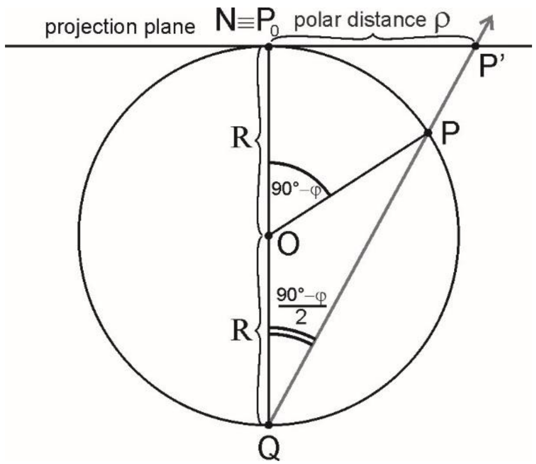
due to the orthogonality of the graticule.

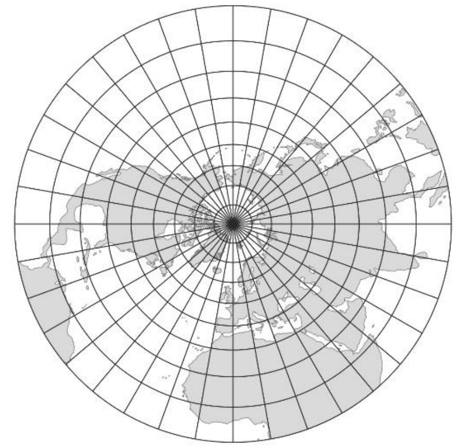


The grayscale level indicates the magnitude of distortions in the normal and oblique azimuthal projections (white shows the negligible distortions)

Stereographic projection of the sphere (1)

Perspective azimuthal projection: the **centre Q of the projection** coincides with one of the poles, and the **map plane** is mostly tangent to the sphere of radius R at the other pole (but it can cut the sphere, too).





Due to the central angle theorem (the **inscribed angle** is always half of the **central angle** belonging to the same circular arc):

$$\rho = 2 \cdot R \cdot \tan\left(\frac{90° - \varphi}{2}\right)$$

Using trigonometric identities:

$$\rho = 2 \cdot R \cdot \frac{\sin\left(\frac{90° - \varphi}{2}\right)}{\cos\left(\frac{90° - \varphi}{2}\right)} = 2 \cdot R \cdot \frac{2 \cdot \sin\left(\frac{90° - \varphi}{2}\right) \cdot \cos\left(\frac{90° - \varphi}{2}\right)}{2 \cdot \cos^2\left(\frac{90° - \varphi}{2}\right) + \sin^2\left(\frac{90° - \varphi}{2}\right) - \sin^2\left(\frac{90° - \varphi}{2}\right)} = 2 \cdot R \cdot \frac{\cos\varphi}{1 + \sin\varphi}$$

Stereographic projection of the sphere (2)

The projection equations:

$$x = 2 \cdot R \cdot \frac{\cos\varphi}{1+\sin\varphi} \cdot \cos(\lambda - \lambda_0)$$

$$y = -2 \cdot R \cdot \frac{\cos\varphi}{1+\sin\varphi} \cdot \sin(\lambda - \lambda_0)$$

The linear scale along the graticule lines:

$$h = \frac{2 \cdot R \cdot \cos\varphi}{(1+\sin\varphi) \cdot R \cdot \cos\varphi} = \frac{2}{1+\sin\varphi}$$

$$k = -\frac{2 \cdot R \cdot [(-\sin\varphi) \cdot (1+\sin\varphi) - \cos\varphi \cdot \cos\varphi]}{(1+\sin\varphi)^2} \cdot \frac{1}{R} = \frac{(-2) \cdot (-\sin\varphi - \sin^2\varphi - \cos^2\varphi)}{(1+\sin\varphi)^2} = \frac{2}{1+\sin\varphi}$$

This distortions depend only on the latitude, so they are constant along the parallel circles.The map graticule is rectangular ($\cot\theta = 0$), so the maximum and minimum linear scale occur in their directions, and the equation *h=k* results in **conformity.** The linear scale *l* in any direction: *l=h=k*, the area scale *p: p=h⋅k=l²*.

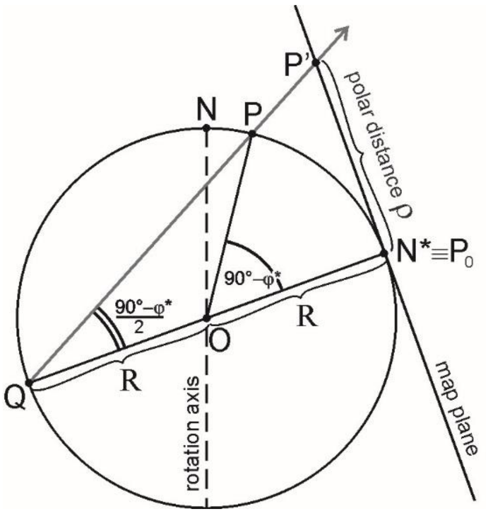
In the **tangent point** of the map plane ($\varphi = 90°$ or $\varphi = -90°$) *l=h=k=1* in other words there is **no distortion** locally in the pole. Moving away from the pole, the distortions are continuously growing, but in the neighbourhood of the pole, they are small.

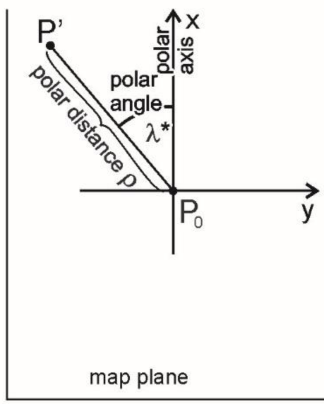
(The plane can **cut** the sphere along a **parallel** which is without distortions.)

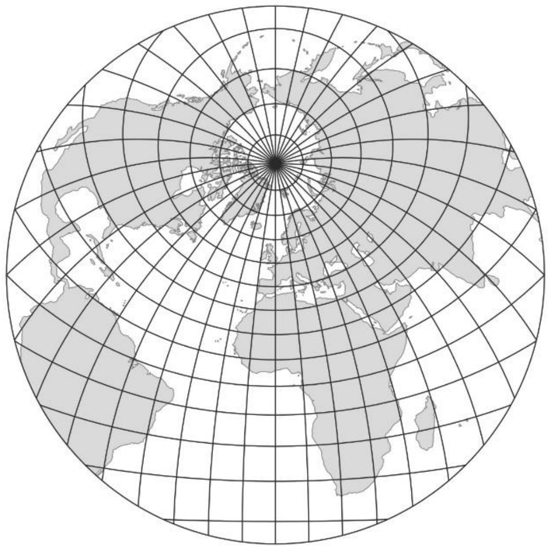
This projection provides advantageous distortion conditions for the **conformal** representation of a circular area about the pole.

Oblique stereographic projection of the sphere

For maps representing approximately circular territories farther from the pole **conformally**, an **oblique stereographic projection** can be applied, with the metapole in the centre of the area to be represented.







The projection equations are referred to the metacoordinates:

$$x = 2 \cdot R \cdot \frac{\cos\varphi^*}{1+\sin\varphi^*} \cdot \cos\lambda^*$$

$$y = -2 \cdot R \cdot \frac{\cos\varphi^*}{1+\sin\varphi^*} \cdot \sin\lambda^*$$

The calculation of the map coordinates x, y from the geographic coordinates φ, λ is carried out by using the **transformation formulae** between the geographic coordinates and metacoordinates φ^*, λ^*.

Conformal azimuthal („stereographic”) projection of the ellipsoid

The azimuthal projections of the ellipsoid are described by the equations

$$x = \rho(\Phi)\cdot\cos(\Lambda - \Lambda_0)$$
$$y = -\rho(\Phi)\cdot\sin(\Lambda - \Lambda_0)$$

The graticule distortions:

$$h = \frac{\sqrt{\left(\frac{\partial x}{\partial \Lambda}\right)^2 + \left(\frac{\partial y}{\partial \Lambda}\right)^2}}{N(\Phi)\cdot\cos\Phi} = \frac{\rho(\Phi)}{N(\Phi)\cdot\cos\Phi} \qquad k = \frac{\sqrt{\left(\frac{\partial x}{\partial \Phi}\right)^2 + \left(\frac{\partial y}{\partial \Phi}\right)^2}}{M(\Phi)} = -\frac{d\rho}{d\Phi}\cdot\frac{1}{M(\Phi)}$$

The map graticule is rectangular ($\cot\theta=0$), the maximum and minimum linear scale occur in their directions, so the equation *h*=*k* results in **conformity**:

$$\frac{\rho(\Phi)}{N(\Phi)\cdot\cos\Phi} = -\frac{d\rho}{d\Phi}\cdot\frac{1}{M(\Phi)}$$

Solving the equation, the radius ρ of mapped parallels can be obtained:

$$\rho = d\cdot\frac{\cos\Phi}{1+\sin\Phi}\cdot\left(\frac{1+e\cdot\sin\phi}{1-e\cdot\sin\phi}\right)^{\frac{e}{2}}$$

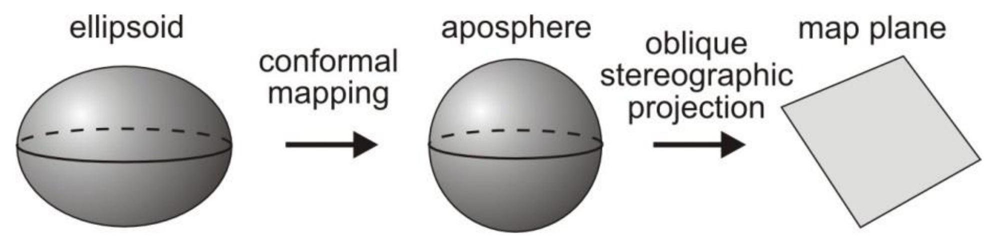
An application of this projection is the Universe Polar Stereographic (**UPS**, not perspective) for the topographic maps of the polar area completing the World map system UTM. The origin is shifted from the pole 2000 km to west (false easting) and 2000 km to south (false northing). An other application: representation of the poles for **World Aeronautical Chart** (scale 1:1 million)

Conformal mapping of the ellipsoid by „double stereographic projection”

An intermediate sphere („aposphere”) will be inserted amongst the ellipsoid and map plane. Two **conformal** mappings will be combined:

- a transformation from the ellipsoid to the aposphere,
- an oblique stereographic projection from the aposphere to the plane.

It is the „**double stereographic projection**”.



The general forward equations created by Thomas:

$$x = 2 \cdot k_0 \cdot \frac{N(\Phi_0) \cdot \cos\Phi_0}{\cos\varphi_0} \cdot \frac{\cos\varphi_0 \cdot \sin\varphi - \sin\varphi_0 \cdot \cos\varphi \cdot \cos(\Lambda - \Lambda_0)}{1 + \sin\varphi_0 \cdot \sin\varphi + \cos\varphi_0 \cdot \cos\varphi \cdot \cos(\Lambda - \Lambda_0)}$$

$$y = 2 \cdot k_0 \cdot \frac{N(\Phi_0) \cdot \cos\Phi_0}{\cos\varphi_0} \cdot \frac{\cos\varphi \cdot \sin(\Lambda - \Lambda_0)}{1 + \sin\varphi_0 \cdot \sin\varphi + \cos\varphi_0 \cdot \cos\varphi \cdot \cos(\Lambda - \Lambda_0)}$$

(k_0 is the linear scale in the central point Φ_0,Λ_0). The formula giving the spherical latitude φ from the ellipsoidal latitude Φ (n=1, κ=1):

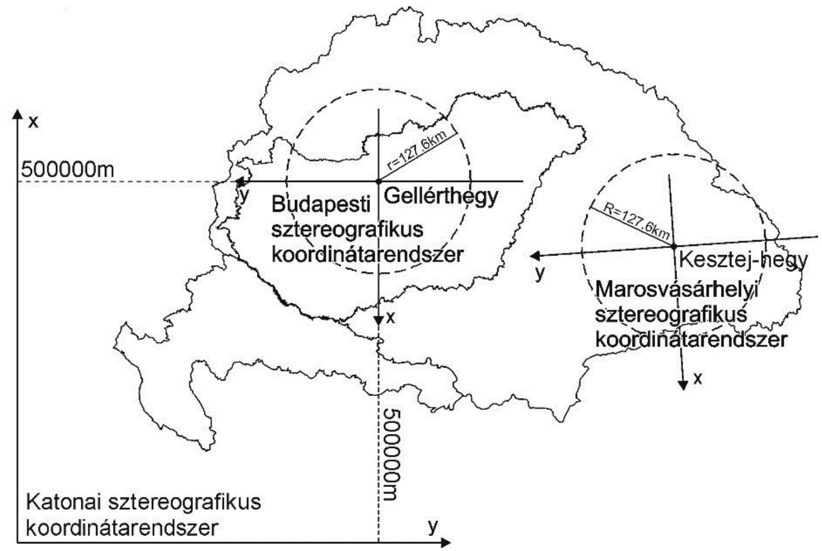
$$\varphi = 2 \cdot \arctan\left[\tan\left(45° + \frac{\Phi}{2}\right) \cdot \left(\frac{1 - e \cdot \sin\Phi}{1 + e \cdot \sin\Phi}\right)^{\frac{e}{2}} \right] - 90°$$

To the **inverse** calculations $\Lambda=\lambda$, Φ by iteration from φ and $R = \dfrac{k_0 \cdot N(\Phi_0) \cdot \cos\Phi_0}{\cos\varphi_0}$

Applications for conformal representation

Applications of the **double stereographic projection**:

- Bessel ellipsoid, Amersfoort datum, „Amersfoort/RD old" projection for the topographic map system in Netherlands
  φ_0=52° 9'22.178", λ_0=5°23'15.5",
  R=6377397.155m; k_0=0.9999079
- Bessel ellipsoid, HD1863 datum,
  „Budapest stereographic" and „military stereographic" projection in Hungary:
  φ_{0Bp}=47°26'21.1372",λ_{0Bp}=0°0'0" (+36°42'53.5733" Ferro); k_0=1.000
  coordinate axes x,y are directed to SW.
  Military system: origin is shifted to SW by 500000m,500000m and axes are turned to NE.



Roussilhe projection: mapping of the ellipsoidal surface onto plane by a power series expansion of a complex function. Its distortion properties are very similar to the same of a double stereographic function. Applications:

- „Stereo70" for the topographic map system of Romania
- „UKLAD" for the topographic map system of Poland

Example: φ_{BP1}=47°25'37.64" λ_{Bp1}=0°0'52.46"; φ^*=? λ^*=? x_{Bp1}=? y_{Bp1}=?
(HD1863, radius of osculating sphere: R=6378512.966m)

Home work: x_{mil}=498654.92m, y_{mil}=501097.61m; Φ_{BP1}=? Λ_{Bp1}=?

Map distortions (2)

Map projections cannot be at the same time both *equal-area* and *conformal*. The so called *aphylactic* projections are neither conformal, nor equal-area. In terms of *application*, both the military and civil topography (inclusive of cadastral cartography) generally use conformal projections, many kinds of thematic maps are created in equal-area projection, and most of the chorographic maps has an aphylactic projection.

Based on the *projection equations* of a map projection, the local distortions at each point on the map can be determined in a few steps.

1. calculation of the *graticule distortions*, namely the linear scale *h* along the parallel, the linear scale *k* along the meridian, and the angle Θ between the images of the graticule lines on the appointed location (see next page):

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and

$$\sin\Theta = \frac{1}{\sqrt{1+\cot^2\Theta}}$$

The scale distortions along the graticule lines in case of ellipsoidal projection:

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and $$b = \frac{\sqrt{h^2 + k^2 + 2 \cdot h \cdot k \cdot \sin\Theta} - \sqrt{h^2 + k^2 - 2 \cdot h \cdot k \cdot \sin\Theta}}{2}$$

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The formula for the area scale p is:

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(equation of the conformity);

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$$a \cdot b = 1$$

(equation of the equivalency);

the *true scale* property at a point of the map in a direction whose angle closed by the first principal direction equals δ:

$$\sqrt{a^2 \cdot cos^2\delta + b^2 \cdot sin^2\delta} = 1$$

Home work:

Ellipsoid-sphere transformation constants: κ=1.0031100083,

n=1.0007197049, e=0.0818205679

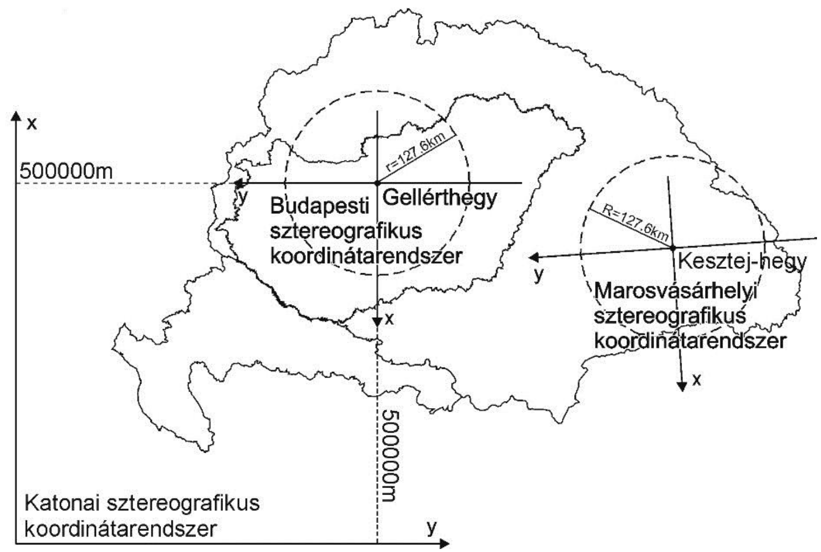
Given the spherical coordinates of a point :

φ=47°25'480.5686", λ=0°0'52.4618" (+19°2'54.8584").

Asked ellipsoidal coordinates Φ and Λ (HD72 datum, IUGG67 ellipsoid).

Applications of the double stereographic projection:

- „Schreiber projection” for the topographic map system in Netherlands
- „Budapest stereographic” and „military stereographic” projection in Hungary: metapole φ_{0Bp}=47°26’21.1372” λ_{0Bp}=0°0’0” (+36°42’53.5733” Ferro); coordinate axes x,y are directed to SW. Military system: origin is shifted to SW by 500000m,500000m and turned to NE



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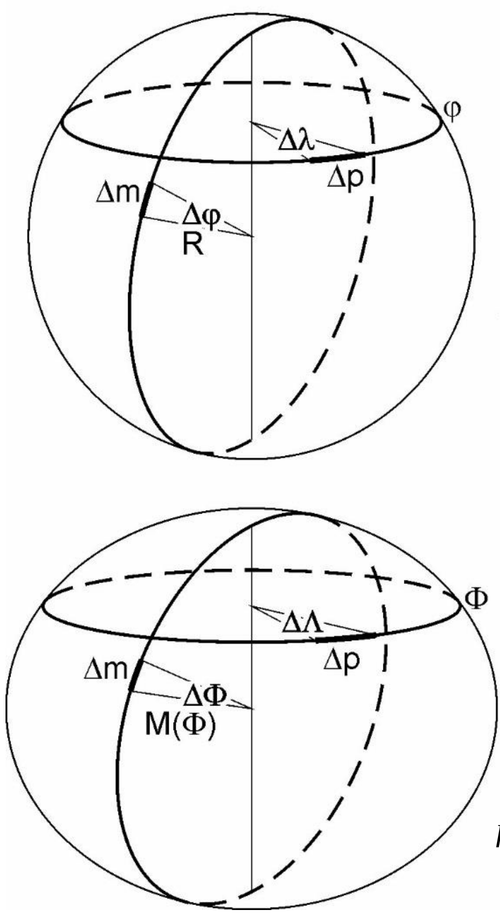
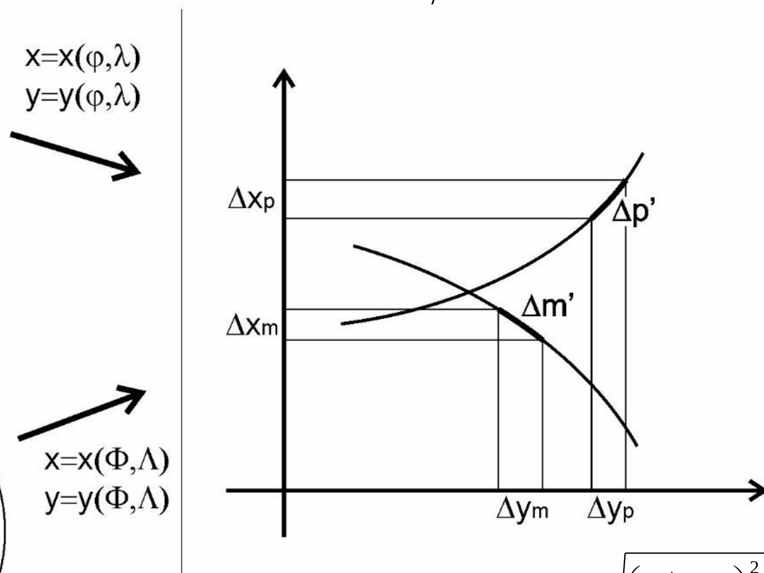
(HD1863, **Bessel:** k=1.003016135, n=1.0007514896, e=0.081696831)
Home work: x_{mil}=498654.92m, y_{mil}=501097.61m; Φ_{BP1}=? Λ_{Bp1}=?

The linear scales along the graticule lines in map projections

$$h=\lim_{\Delta\lambda\to 0}\frac{\Delta p'}{\Delta p}=\lim_{\Delta\lambda\to 0}\frac{\sqrt{(\Delta x_p)^2+(\Delta y_p)^2}}{R\cdot\cos\varphi\cdot arc\Delta\lambda}=\lim_{\Delta\lambda\to 0}\frac{\sqrt{\left(\frac{\Delta x_p}{arc\Delta\lambda}\right)^2+\left(\frac{\Delta y_p}{arc\Delta\lambda}\right)^2}\cdot arc\Delta\lambda}{R\cdot\cos\varphi\cdot arc\Delta\lambda}=\frac{\sqrt{\left(\frac{\partial x}{\partial\lambda}\right)^2+\left(\frac{\partial y}{\partial\lambda}\right)^2}}{R\cdot\cos\varphi}$$

$$k=\lim_{\Delta\varphi\to 0}\frac{\Delta m'}{\Delta m}=\lim_{\Delta\varphi\to 0}\frac{\sqrt{(\Delta x_m)^2+(\Delta y_m)^2}}{R\cdot arc\Delta\varphi}=\lim_{\Delta\varphi\to 0}\frac{\sqrt{\left(\frac{\Delta x_m}{arc\Delta\varphi}\right)^2+\left(\frac{\Delta y_m}{arc\Delta\varphi}\right)^2}\cdot arc\Delta\varphi}{R\cdot arc\Delta\varphi}=\frac{\sqrt{\left(\frac{\partial x}{\partial\varphi}\right)^2+\left(\frac{\partial y}{\partial\varphi}\right)^2}}{R}$$





$$h=\lim_{\Delta\Lambda\to 0}\frac{\Delta p'}{\Delta p}=\lim_{\Delta\Lambda\to 0}\frac{\sqrt{(\Delta x_p)^2+(\Delta y_p)^2}}{N(\Phi)\cdot\cos\Phi\cdot arc\Delta\Lambda}=\lim_{\Delta\Lambda\to 0}\frac{\sqrt{\left(\frac{\Delta x_p}{arc\Delta\Lambda}\right)^2+\left(\frac{\Delta y_p}{arc\Delta\Lambda}\right)^2}\cdot arc\Delta\Lambda}{N(\Phi)\cdot\cos\Phi\cdot arc\Delta\Lambda}=\frac{\sqrt{\left(\frac{\partial x}{\partial\Lambda}\right)^2+\left(\frac{\partial y}{\partial\Lambda}\right)^2}}{N(\Phi)\cdot\cos\Phi}$$

$$k=\lim_{\Delta\Phi\to 0}\frac{\Delta m'}{\Delta m}=\lim_{\Delta\Phi\to 0}\frac{\sqrt{(\Delta x_m)^2+(\Delta y_m)^2}}{M(\Phi)\cdot arc\Delta\Phi}=\lim_{\Delta\Phi\to 0}\frac{\sqrt{\left(\frac{\Delta x_m}{arc\Delta\Phi}\right)^2+\left(\frac{\Delta y_m}{arc\Delta\Phi}\right)^2}\cdot arc\Delta\Phi}{M(\Phi)\cdot arc\Delta\Phi}=\frac{\sqrt{\left(\frac{\partial x}{\partial\Phi}\right)^2+\left(\frac{\partial y}{\partial\Phi}\right)^2}}{M(\Phi)}$$