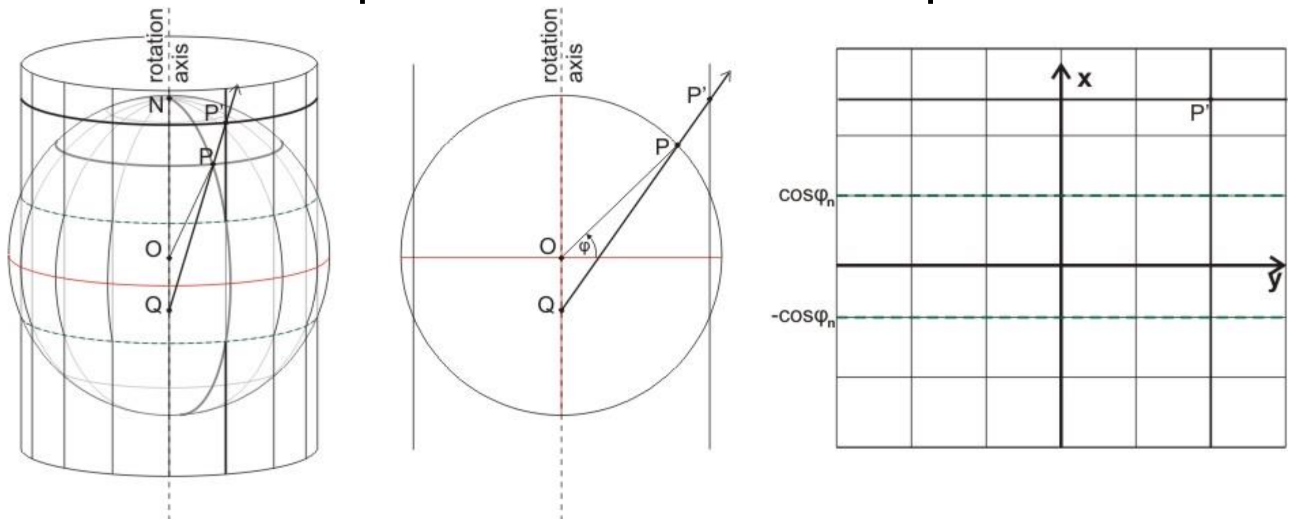
Lesson 8

- Cylindrical map coordinate systems
- Mercator (conformal cylindrical) projection of the sphere
- Oblique Mercator projection
- Conformal cylindrical projection of the ellipsoid
- Double Mercator projection

Cylindrical map projections in general (1)

Central perspective projection from a spherical surface onto a superficies of a **cylinder of revolution** by projectors: in order to reflection symmetry, the centre of projection is located on the rotation axis of the earth sphere, and the rotation axes of the superficies and the earth sphere coincide.

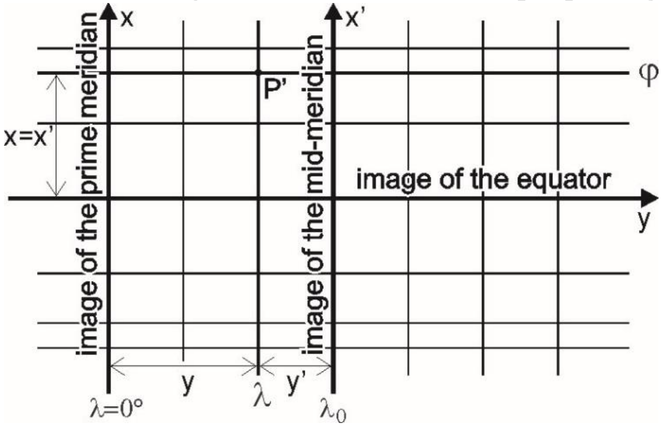


After cutting the superficies by one of its generators and unfolding it into a plane, the image of the graticule passes over a planar rectangular grid:

a) the images of the parallels are **parallel straight lines**, b) the images of the meridians are **parallel straight lines**, too, c) they are **perpendicular** to each other, and d) the distance between the mapped meridians is **proportional** to the correspondent **longitude difference**.

Cylindrical map projections in general (2)

If the map graticule (or metagraticule) has the previous properties then it is called **cylindrical map projection.**



The graticule (metagraticule) has generally two axes of reflection symmetry. The horizontal one usually coincides with the axis y which is mostly the image of the equator.

Rectangular coordinates of the normal version:

$$x = x(\varphi)$$

$$y = c \cdot R \cdot arc(\lambda - \lambda_0)$$

where x(φ) is a strictly increasing and possibly odd function, and the coefficient c depends on the true scale parallel φ_s, practically $c=\cos\varphi_s$.

The graticule distortions are:

$$h = \frac{c}{\cos\varphi} = \frac{\cos\varphi_s}{\cos\varphi} \qquad k = \frac{dx}{d\varphi} \cdot \frac{1}{R} \qquad \cot\theta = 0$$

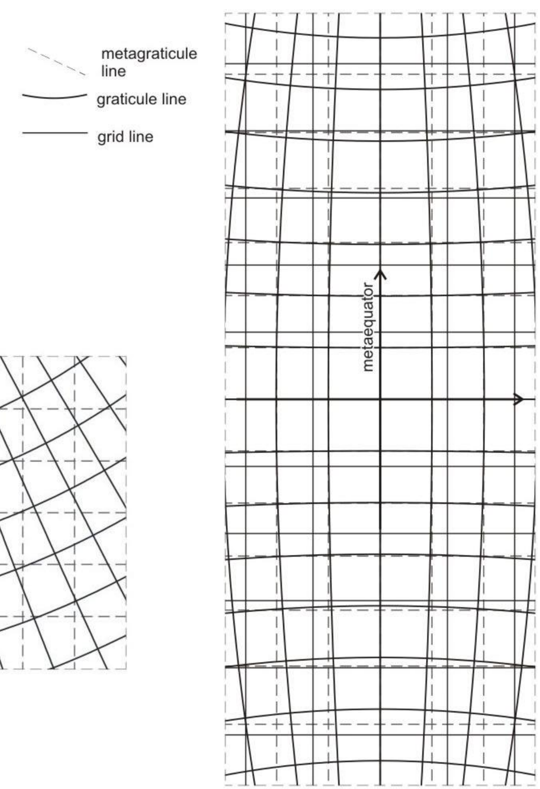
Cylindrical map projections in general (3)

In the case of transverse or oblique version:

$$x = x(\varphi^*) \qquad y = c \cdot R \cdot arc\,\lambda^*$$

where $c=\cos(\varphi_s^*)$
(φ_s^* is the true scale metaparallel).

transverse Mercator projection



Ellipsoidal formulae:

$$x = x(\Phi)$$

$$y = N(\Phi_s) \cdot \cos\Phi_s \cdot arc(\Lambda - \Lambda_0)$$

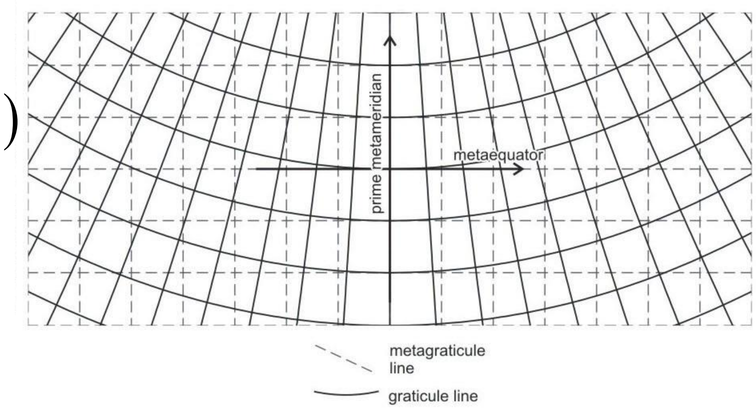
Graticule distortions:

$$h = \frac{N(\Phi_s) \cdot \cos\Phi_s}{N(\Phi) \cdot \cos\Phi}$$

$$k = \frac{dx}{d\Phi} \cdot \frac{1}{M(\Phi)}$$

and $\cot\Theta = 0$.

oblique Mercator projection



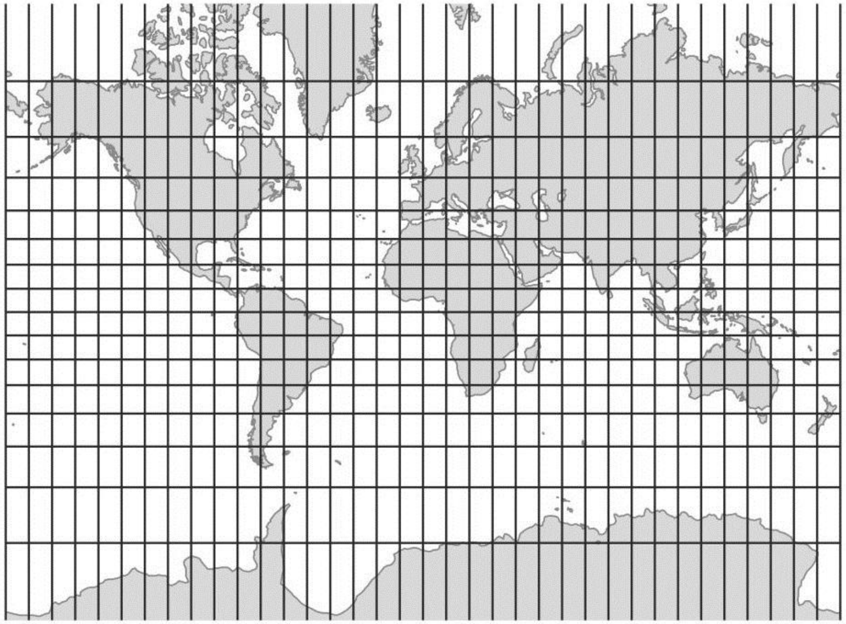
Mercator's conformal cylindrical projection of the sphere (1)

Equation of conformity *h=k* is in this case: $\frac{\cos\varphi_s}{\cos\varphi} = \frac{dx}{d\varphi}\cdot\frac{1}{R}$

Consequently $x = R\cdot\cos\varphi_s\cdot\int\frac{d\varphi}{\cos\varphi} = R\cdot\cos\varphi_s\cdot\ln\tan\left(45^\circ+\frac{\varphi}{2}\right)$ (constant of integr.=0)

The coordinate x will be transformed:

$$\left[\tan^2\left(45^\circ+\frac{\varphi}{2}\right)\right]^{\frac{1}{2}} = \left[\frac{\left(\frac{\sqrt{2}}{2}\cdot\cos\frac{\varphi}{2}+\frac{\sqrt{2}}{2}\cdot\sin\frac{\varphi}{2}\right)^2}{\left(\frac{\sqrt{2}}{2}\cdot\cos\frac{\varphi}{2}-\frac{\sqrt{2}}{2}\cdot\sin\frac{\varphi}{2}\right)^2}\right]^{\frac{1}{2}} = \left[\frac{1+2\cdot\sin\frac{\varphi}{2}\cdot\cos\frac{\varphi}{2}}{1-2\cdot\sin\frac{\varphi}{2}\cdot\cos\frac{\varphi}{2}}\right]^{\frac{1}{2}} = \left[\frac{1+\sin\varphi}{1-\sin\varphi}\right]^{\frac{1}{2}}$$



Hence the rectangular coordinates:

$$x = R\cdot\cos\varphi_s\cdot\ln\tan\left(45^\circ+\frac{\varphi}{2}\right) = \frac{R}{2}\cdot\cos\varphi_s\cdot\ln\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)$$

$$y = R\cdot\cos\varphi_s\cdot arc\lambda$$

The poles can not be represented.

The inverse projection equations:

$$\varphi = 2\cdot\arctan\left[\exp\left(\frac{x}{R\cdot\cos\varphi_s}\right)\right]-90^\circ$$

$$arc\lambda = \frac{y}{R\cdot\cos\varphi_s}$$

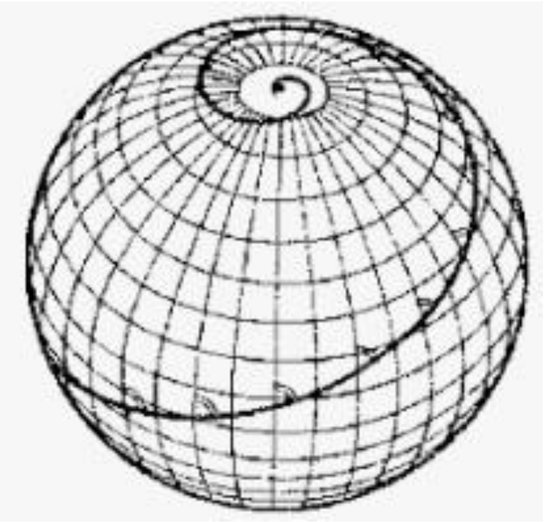
Mercator’s conformal cylindrical projection of the sphere (2)

The linear scales: $l = h = k = \frac{cos\varphi_S}{cos\varphi}$; the area scales: $p = h \cdot k = \frac{cos^2\varphi_S}{cos^2\varphi}$

There is not any distortion along the standard parallels. In the case of $\pm\varphi_S$=0 the equator is free of distortions. Far from the equator the linear and area scales grow strongly. The images of the poles are infinitely far away.

The loxodrome on the map of Mercator’s projection

A *loxodrome* (rhumb line) is a line on the surface of the sphere, which has the same azimuth α in its every point (α=constant).
Thus the meridians (α=0° or α=180°) and the parallels (α=90° or α=270°) are trivial loxodromes.
The loxodromes with a differing azimuth are *spherical spirals* leading from one pole to the other.



If the azimuth $\alpha\neq$90° then to each value of the latitude φ belongs only one longitude λ. Asked the single-valued function $\lambda=\lambda(\varphi)$.

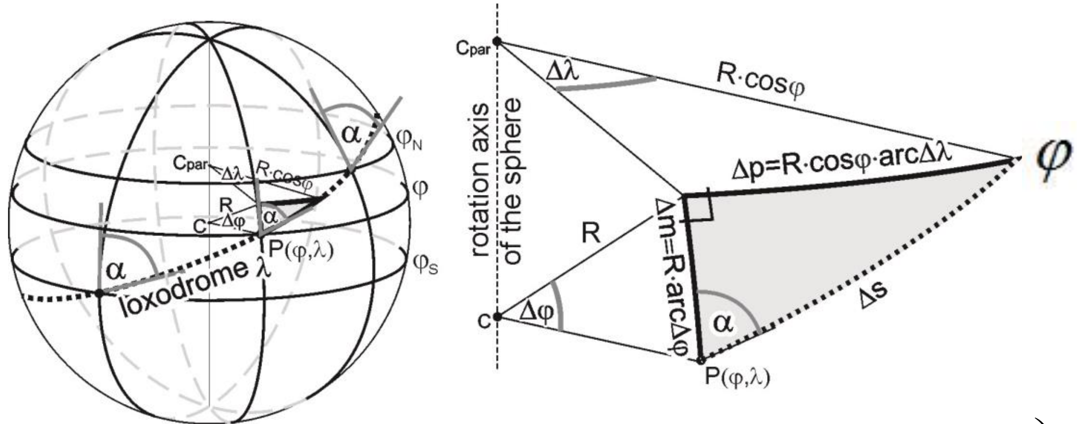
The coordinates φ and λ of the point P on the loxodrome arc will be changed by Δφ and Δλ. This establishes a small figure which is almost a small planar right triangle with the loxodrome arc Δs ("hypotenuse") as well as parallel arc R∗cosφ∗arcΔλ ("leg") and meridian arc R∗arcΔφ ("leg").

Then approximately:

$$\tan\alpha \approx \frac{R \cdot \cos\varphi \cdot arc\Delta\lambda}{R \cdot arc\Delta\varphi}$$

rearranged:

$$arc\Delta\lambda \approx \frac{\tan\alpha \cdot arc\Delta\varphi}{\cos\varphi}$$



Partitioning the loxodrome arc into small parts, the sum converges to the integral:

$$\int d\lambda = \int \frac{\tan\alpha}{\cos\varphi} d\varphi = \tan\alpha \cdot \int \frac{d\varphi}{\cos\varphi}$$

and the *function* λ=λ(φ): $\lambda = \tan\alpha \cdot \ln\tan\left(45^\circ + \frac{\varphi}{2}\right) + C$

where, if crossing φ_0,λ_0, the constant C of integration: $C = \lambda_0 - \tan\alpha \cdot \ln\tan\left(45^\circ + \frac{\varphi_0}{2}\right)$

If $\varphi \to \pm 90^\circ$ then $\lambda \to \pm\infty$, so φ=φ(λ) is a multivalued function.

The azimuth α of a loxodrome crossing the points φ_1,λ_1 and φ_2,λ_2 :

$$\tan\alpha = \frac{\lambda_2 - \lambda_1}{\ln\tan\left(45^\circ + \frac{\varphi_2}{2}\right) - \ln\tan\left(45^\circ + \frac{\varphi_1}{2}\right)}$$

The *length* of a loxodrome arc between the latitudes φ_S and φ_N: $\cos\alpha \approx \frac{R \cdot arc\Delta\varphi}{\Delta s}$,

rearranged: $\Delta s \approx \frac{R \cdot arc\Delta\varphi}{\cos\alpha}$

The sum converges to the integral: $\int ds = R \cdot \int_{\varphi_S}^{\varphi_N} \frac{d\varphi}{\cos\alpha}$ and finally $s = R \cdot \frac{|arc\varphi_N - arc\varphi_S|}{\cos\alpha}$

Mercator's conformal cylindrical projection of the sphere (4)

If the loxodrome of azimuth α passes through the spherical point P_0 (φ_0,λ_0), whose image is the point $P'(x_0,y_0)$ then its equation is:

$$\lambda - \lambda_0 = \tan\alpha \cdot \left[\ln\tan\left(45° + \frac{\varphi}{2}\right) - \ln\tan\left(45° + \frac{\varphi_0}{2}\right) \right]$$

Multiplying it by $R \cdot \cos\varphi_s$ with an arbitrary latitude of φ_s:

$$R \cdot \cos\varphi_s \cdot \lambda - R \cdot \cos\varphi_s \cdot \lambda_0 = \tan\alpha \cdot \left[R \cdot \cos\varphi_s \cdot \ln\tan\left(45° + \frac{\varphi}{2}\right) - R \cdot \cos\varphi_s \cdot \ln\tan\left(45° + \frac{\varphi_0}{2}\right) \right]$$

and applying the projection equations *x* and *y* of the Mercator projection with the true scale parallel of φ_s, the formula

$$y - y_0 = \tan\alpha \cdot (x - x_0)$$

will be got which is an equation of a straight line crossing the map point $P_0'(x_0,y_0)$ assigned to the Earth point $P_0(\varphi_0,\lambda_0)$. This straight line and the vertical axis *x* include an angle α. It proves that the spherical loxodromes are represented by straight lines on those charts which were mapped by the Mercator's projection.

The shipping on the oceans which preferred the pathes along loxodromes from the XVII century, used Mercator's maps to the navigation.

Examples for loxodrome and orthodrome in Mercator projection

Mercator's conformal cylindrical projection of the ellipsoid (1)

Equation of the conformity is: *h=k* which is detailed:

$$\frac{N(\Phi_s)\cdot\cos\Phi_s}{N(\Phi)\cdot\cos\Phi}=\frac{dx}{d\Phi}\cdot\frac{1}{M(\Phi)}$$

that is

$$\frac{dx}{d\Phi}=\frac{M(\Phi)\cdot N(\Phi_s)\cdot\cos\Phi_s}{N(\Phi)\cdot\cos\Phi}=N(\Phi_s)\cdot\cos\Phi_s\cdot\frac{(1-e^2)}{(1-e^2\cdot\sin^2\Phi)\cdot\cos\Phi}$$

It means that

$$x=\int N(\Phi_s)\cdot\Phi_s\cdot\frac{(1-e^2)}{(1-e^2\cdot\sin^2\Phi)\cdot\cos\Phi}\,d\Phi=N(\Phi_s)\cdot\Phi_s\cdot\int\left(\frac{1}{\cos\Phi}-\frac{e^2\cdot\cos^2\Phi}{1-e^2\cdot\sin^2\Phi}\right)d\Phi$$

Carrying out the integration (see Lesson_6, slide 4-5):

$$x=N(\Phi_s)\cdot\cos\Phi_s\cdot\ln\left[\tan\left(45°+\frac{\Phi}{2}\right)\cdot\left(\frac{1-e\cdot\sin\Phi}{1+e\cdot\sin\Phi}\right)^{\frac{e}{2}}\right]$$

and as usual:

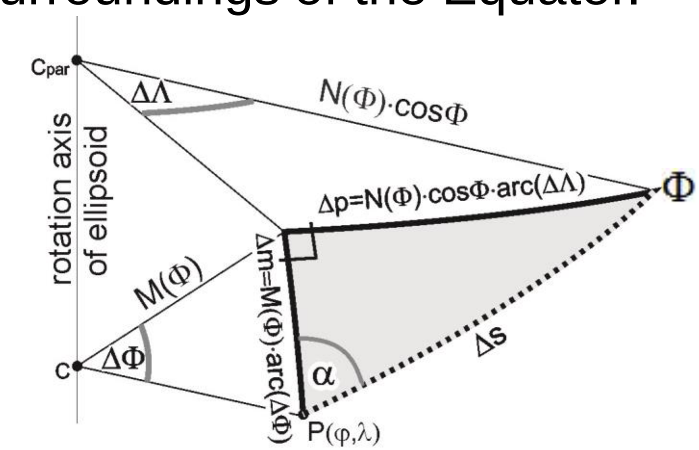
$$y=N(\Phi_s)\cdot\cos\Phi_s\cdot arc(\Lambda-\Lambda_0)$$

Applied for large scale maps representing the surroundings of the Equator.

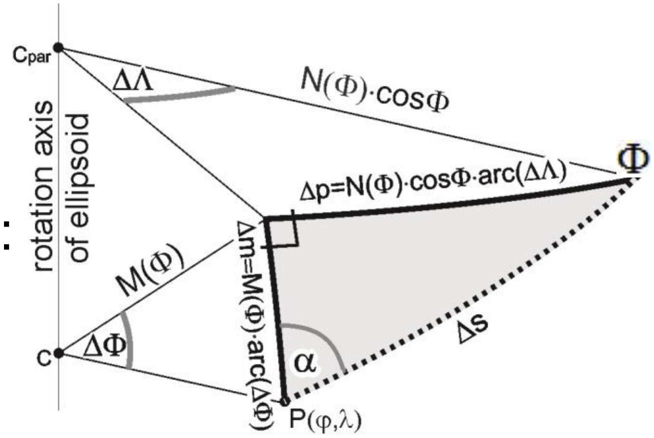
Loxodrome on the ellipsoid

Derivation of the equation, similarly to the spherical one, based on the small right triangle:

$$\tan\alpha\approx\frac{N(\Phi)\cdot\cos\Phi\cdot\Delta\Lambda}{M(\Phi)\cdot\Delta\Phi}$$



Mercator's conformal cylindrical projection of the ellipsoid (2)



rearranged:

$$\Delta\Lambda \approx \tan\alpha \cdot \frac{M(\Phi)\cdot\Delta\Phi}{N(\Phi)\cdot\cos\Phi} = \tan\alpha \cdot \frac{(1-e^2)}{(1-e^2\cdot\sin^2\Phi)\cdot\cos\Phi}\cdot\Delta\Phi$$

Taking sum of both sides and refining the partition:

$$\int d\Lambda = \tan\alpha \cdot \int \frac{(1-e^2)}{(1-e^2\cdot\sin^2\Phi)\cdot\cos\Phi}\, d\Phi$$

Equation of the loxodrome crossing $P_0(\Phi_0,\Lambda_0)$:

$$\Lambda - \Lambda_0 = \tan\alpha \cdot \left\{ \ln\left[\tan\left(45^\circ + \frac{\Phi}{2}\right)\cdot\left(\frac{1-e\cdot\sin\Phi}{1+e\cdot\sin\Phi}\right)^{\frac{e}{2}}\right] - \ln\left[\tan\left(45^\circ + \frac{\Phi_0}{2}\right)\cdot\left(\frac{1-e\cdot\sin\Phi_0}{1+e\cdot\sin\Phi_0}\right)^{\frac{e}{2}}\right]\right\}$$

Multiplying both sides by $N(\Phi_s)*\cos\Phi_s$:

$$N(\Phi_s)\cdot\cos\Phi_s\cdot(\Lambda-\Lambda_0) = \tan\alpha\cdot\left\{ N(\Phi_s)\cdot\cos\Phi_s\cdot\ln\left[\tan\left(45^\circ+\frac{\Phi}{2}\right)\cdot\left(\frac{1-e\cdot\sin\Phi}{1+e\cdot\sin\Phi}\right)^{\frac{e}{2}}\right] - N(\Phi_s)\cdot\cos\Phi_s\cdot\ln\left[\tan\left(45^\circ+\frac{\Phi_0}{2}\right)\cdot\left(\frac{1-e\cdot\sin\Phi_0}{1+e\cdot\sin\Phi_0}\right)^{\frac{e}{2}}\right]\right\}$$

Applying the Mercator projection equations x, y: $y - y_0 = \tan\alpha * (x - x_0)$

which is equation of a *straight line* crossing $P_0'(x_0,y_0)$ on the map and forming an angle α with the axis x.

Arc length of the loxodrome between Φ_S and Φ_N:

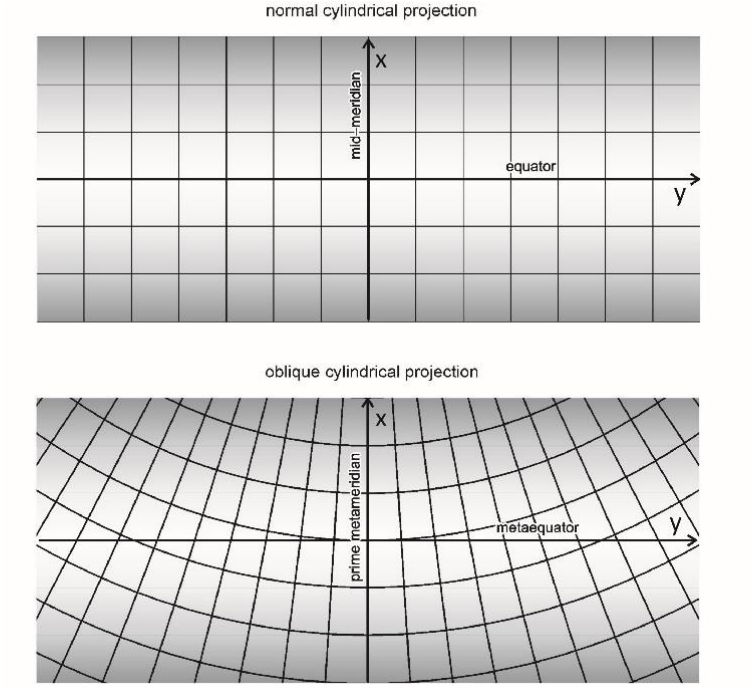
from the figure above: $\cos\alpha \approx \frac{M(\Phi)\cdot\Delta\Phi}{\Delta s}$; rearranged: $\Delta s \approx \frac{M(\Phi)\cdot\Delta\Phi}{\cos\alpha}$

summarizing $\sum_i \Delta s_i \approx \frac{1}{\cos\alpha}\cdot\sum_i M(\emptyset_i)\cdot\Delta\emptyset_i$; the arc length s $s = \int ds = \frac{1}{\cos\alpha}\cdot\left|\int_{\emptyset_S}^{\emptyset_N} M(\emptyset)\, d\emptyset\right|$

Oblique cylindrical map projections

If the area to be represented expands along a great circle of a sphere, then this great circle can be taken as the metaequator of a metacoordinate system, and the projection equations are related to the metacoordinates.

By the oblique cylindrical projections the distortions increase with the distance from the metaequator while they are negligible near the metaequator.



The grayscale level indicates the magnitude of the map distortions in the normal and oblique cylindrical projections (white shows negligible distortions)

Oblique conformal cylindrical map projections (1)

A conformal cylindrical projection should be referred to the metacoordinates φ*, λ*:

$$x = R \cdot \cos\varphi_s * \cdot \ln\tan\left(45° + \frac{\varphi *}{2}\right) = \frac{R}{2} \cdot \cos\varphi_s * \cdot \ln\left(\frac{1+\sin\varphi *}{1-\sin\varphi *}\right)$$

$$y = R \cdot \cos\varphi_s * \cdot arc\,\lambda *$$

Using the metacoordinate transformations

$$\sin\varphi* = \cos\varphi_K \cdot \sin\varphi - \sin\varphi_K \cdot \cos\varphi \cdot \cos(\lambda - \lambda_K)$$

and the formula for $\tan\lambda*$ deduced from the law of sines and the cosine rule for sides:

$$\tan\lambda* = \frac{\sin\lambda *}{\cos\lambda *} = \frac{\sin(\lambda - \lambda_K)}{\sin\varphi_K \cdot \tan\varphi + \cos\varphi_K \cdot \cos(\lambda - \lambda_K)}$$

The rectangular coordinates are:

$$x = \frac{R}{2} \cdot \cos\varphi*_s \cdot \ln\left(\frac{1+\sin\varphi*}{1-\sin\varphi*}\right) + x_0 = \frac{R}{2} \cdot \cos\varphi*_s \cdot \ln\left(\frac{1+\cos\varphi_K \cdot \sin\varphi - \sin\varphi_K \cdot \cos\varphi \cdot \cos(\lambda - \lambda_K)}{1-\cos\varphi_K \cdot \sin\varphi + \sin\varphi_K \cdot \cos\varphi \cdot \cos(\lambda - \lambda_K)}\right) + x_0$$

$$y = R \cdot \cos\varphi*_s \cdot \arctan\left(\frac{\sin(\lambda - \lambda_K)}{\tan\varphi \cdot \sin\varphi_K + \cos\varphi_K \cdot \cos(\lambda - \lambda_K)}\right) + y_0$$

where φ_s* is the true scale metaparallel, and P_K (φ_K, λ_K) is the intersection of the metaequator and the prime metameridian, furthermore x_0, y_0 are the planar translation coordinates of the origin („false northing”, „false easting”).

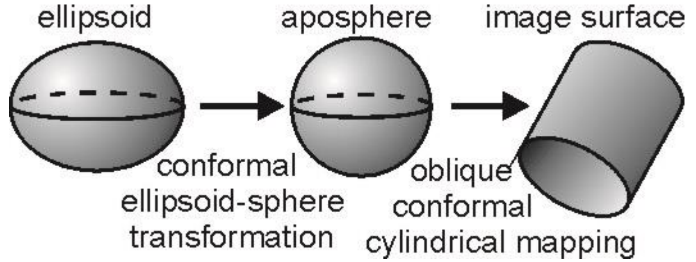
Oblique conformal cylindrical map projections (2)

If $c=\cos(\varphi_s^*)<1$ then the projection is called „reduced", and c provides the measure of the reduction. The earth coordinates can be calculated by reverse use of projection equations:

$$\varphi^* = 2\cdot\arctan\left[\exp\left(\frac{(x-x_0)}{R\cdot\cos\varphi^*_s}\right)\right]-90^\circ \qquad arc(\lambda^*)=\frac{y-y_0}{R\cdot\cos(\varphi^*_s)}$$

then from the metacoordinates into geographic coordinates: $\varphi^*,\lambda^* \rightarrow \varphi,\lambda.$

Mapping the ellipsoid by a double Mercator projection:



a linking of two mappings (a conformal ellipsoid-sphere transformation and an oblique conformal cylindrical projection from the intermediate sphere to the plane).

Applications:

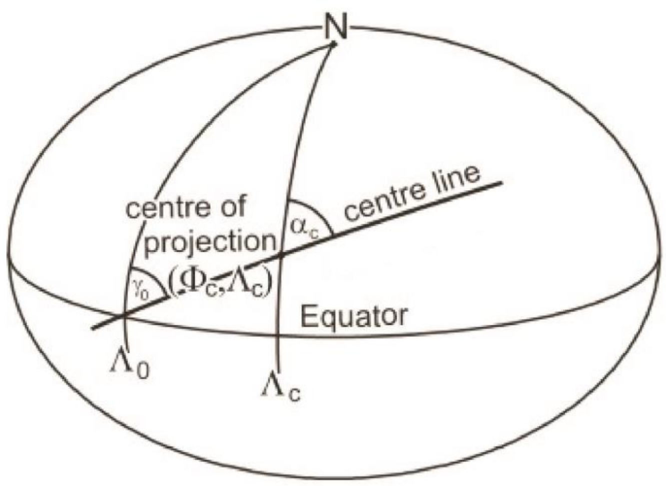
- Rosenmund projection (1903) for the topographic mapping of Switzerland („Swiss Oblique Mercator projection", c=1)
- Projection of Fasching (1909) for the cadastral maps of Hungary (c=1)
- EOV (1975) for the civil map systems of Hungary (c=0.99993).

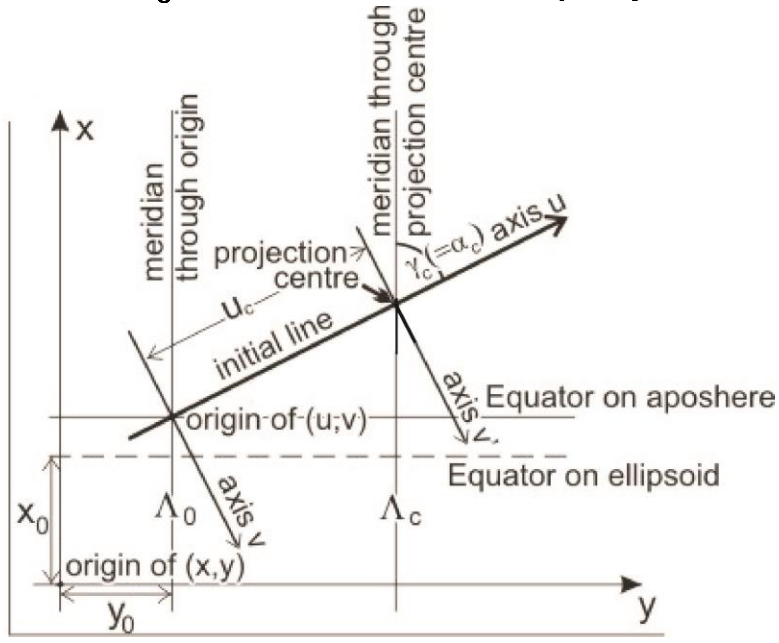
Oblique conformal cylindrical map projections (3)

Mapping the ellipsoid by a Hotine oblique Mercator projection

It is actually a double projection from the ellipsoid onto the plane, where the coordinates of the intermediate surface (not a sphere) do not appear during the calculation, therefore it can be considered as a direct mapping.

This conformal mapping assigns a straight line („initial line") of the map to an ellipsoidal line („centre line"). Given the centre of the projection Φ_C,Λ_C located on the centre line, the azimuth α_c of the centre line taken in the centre of projection, and the scale distortion k_c at the centre of projection.





Home work

Given the map coordinates of P': X=263693.08m and Y=468839.43m in an oblique conformal cylindrical projection, where R=6379743m, $\cos(\varphi_s^*)$=0.99993, x_0=400000m, y_0=650000m, φ_K=47°06'0.0", λ_K=0°.

Asked:

1. the metacoordinates φ^*, λ^* and the spherical coordinates φ, λ of the point P;

2. the linear scale *l* and the area scale *p* at P in normal case (related to φ) and in oblique case (related to φ^*);

3. the azimuth a of the spherical loxodrome connecting the points K and P, and the length of this loxodrome arc.

4. the length of the orthodrome arc connecting K and P.

Extra credit example:

5. the ellipsoidal coordinates Φ,Λ of the point P to be calculated by the iterative sphere-ellipsoid transformation (n=1.0007197049, κ=1.003110007693, e=0.08182056794) in three steps, and the ellipsoidal coordinates of the point K.

6. the azimuth and the length of the ellipsoidal loxodrome arc connecting K and P.