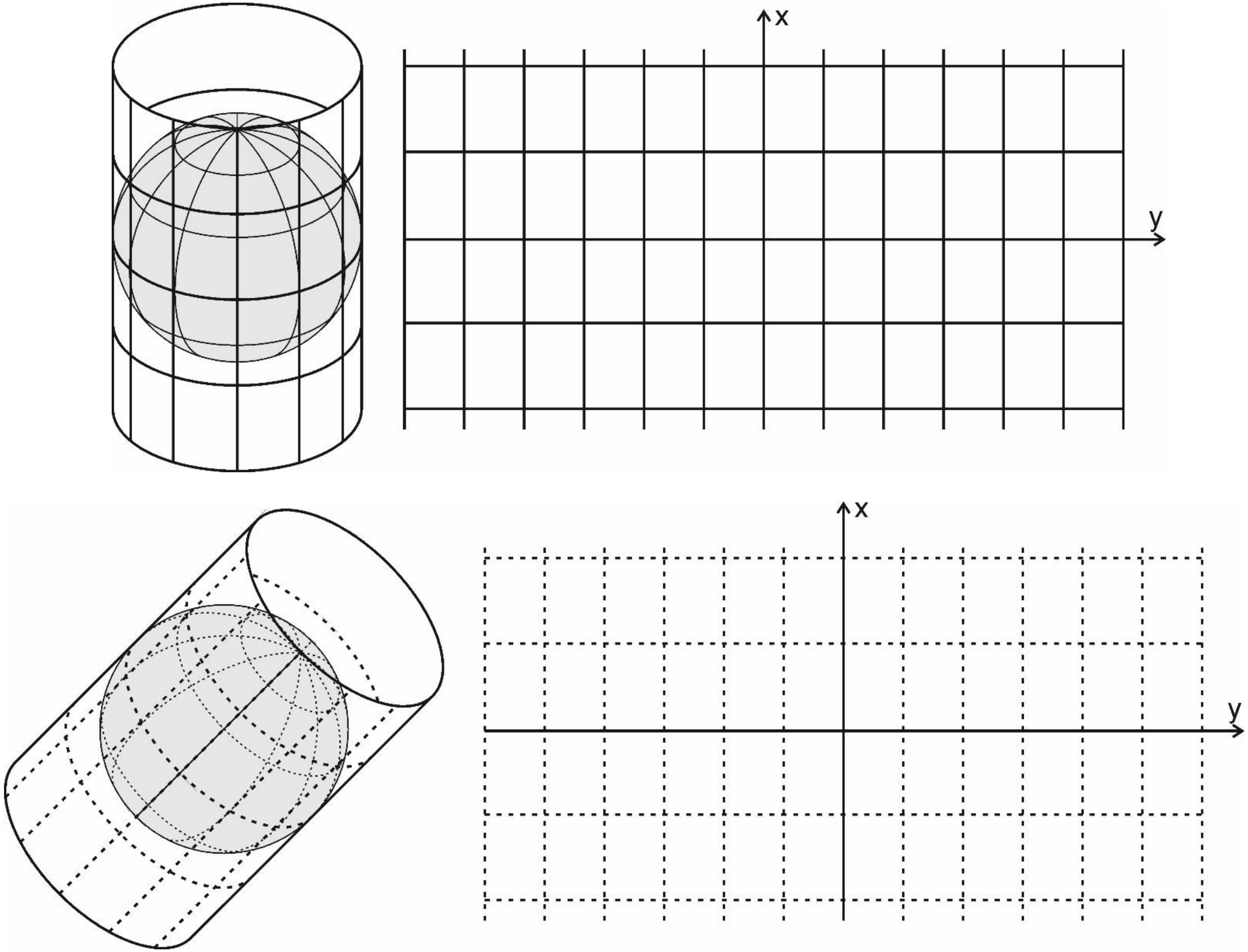
Lesson_9

- Transverse cylindrical map projections
- Cassini projection, Cassini-Soldner projection
- Transverse Mercator projection
- Gauss-Krüger projection and UTM projection
- World map systems using the Gauss-Krüger and UTM projections



Cylindrical projections: the distance between the (meta-) meridians depends on the true scale (meta-) parallel φ_s (φ_s^*), while the distance between the (meta-) parallels depends on the projection equation $x(\varphi)$ $\left(x(\varphi^*)\right)$. On the (conformal) Mercator maps, the distances of parallels grow moving away from the equator.

Equidistant cylindrical projection of the sphere (Plate carrée)

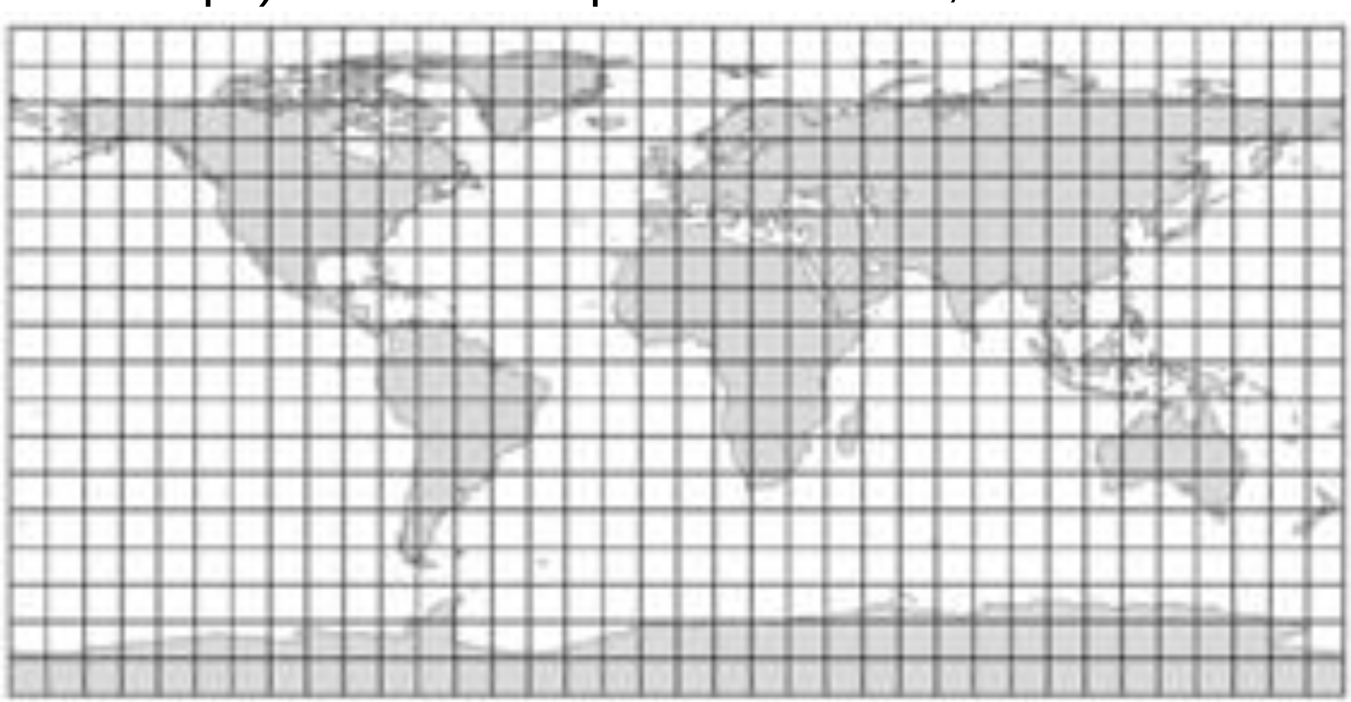
The *meridians are true scale*, in equation: k=1, that is $\frac{dx}{d\varphi} \cdot \frac{1}{R} = 1$

The projection equations:

$x = R \cdot arc\varphi$ and as usual $y = R \cdot \cos(\varphi_s) \cdot arc(\lambda - \lambda_K)$

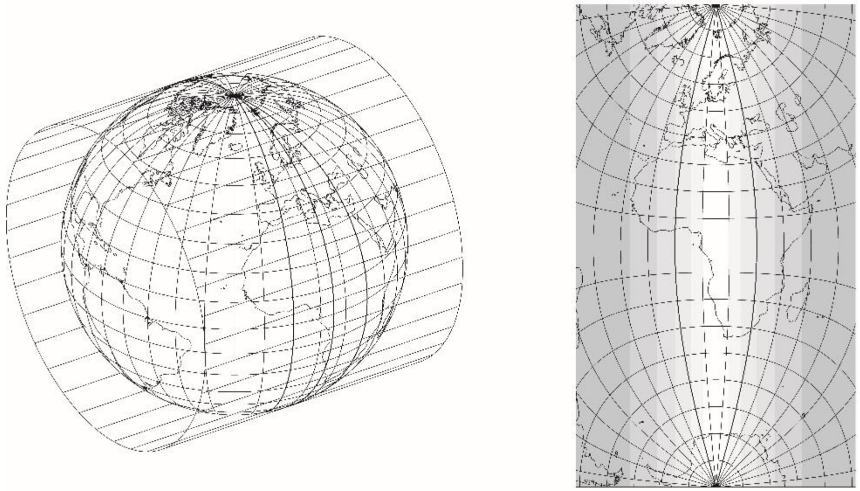
where either the equator (φ_s=0°) or the parallels $\pm\varphi_s$ ($\neq$0°) are true scale.

Neither conformal, nor equal-area, so it is an *aphylactic* projection, applied mainly for small scale chorographic world maps (or e.g. it is the best choice for time zone maps). It has an ellipsoidal version, too.



Transverse cylindrical projections of the sphere (1)

The metapole is situated on the equator, the metaequator coincides with the mid-meridian (denoted by λ_K) of the territory to be represented. This territory is mostly a *spherical lune* bounded by two meridians.



The grayscale level indicates the magnitude of the map distortions in the transverse cylindrical projections (white shows negligible distortions)

The mid-meridian λ_K and the equator are mapped onto *straight lines perpendicular* to each other, which are the axes of symmetry of the graticule, and they can be chosen as coordinate axes. These projections are usually applied for mapping long and relatively narrow areas lying along a meridian, where there are favourable distortions. Either this meridian λ_K (metaequator, $\varphi^*_s=0°$), or two appointed metaparallels $\pm\varphi^*_s$ are true scale, depending on the width of the area to be represented.

Transverse cylindrical projections of the sphere (2)

The projection equations are referred to the metacoordinates φ*, λ*:

$x = x(\varphi^*)$

$y = R * \cos(\varphi^*_s) * arc(\lambda^*)$

The transformation formulae are, substituting 0° into φ_K:

$$\varphi^* = \arcsin\left[\cos\varphi \cdot \sin(\lambda - \lambda_K)\right]$$

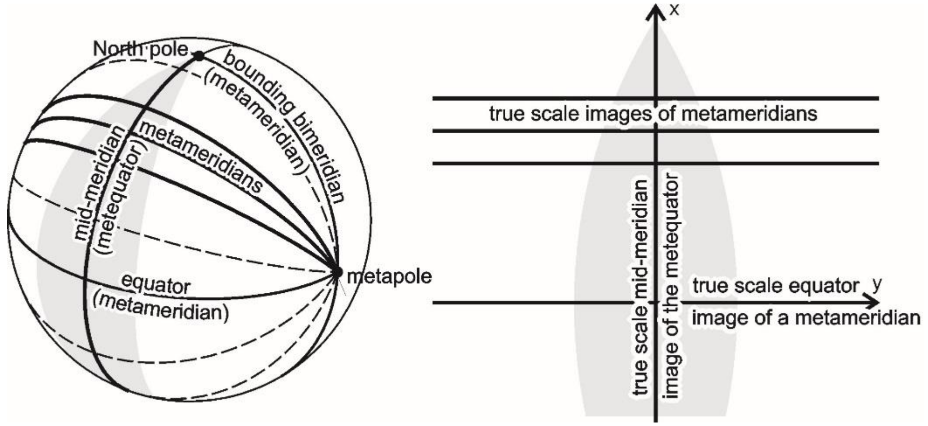
$$\lambda^* = \arcsin\left[\frac{\sin\varphi}{\sqrt{1 - \cos^2\varphi \cdot \sin^2(\lambda - \lambda_K)}}\right]$$

By φ^*_s=0°, it is the *transverse equidistant cylindrical projection* (Cassini projection), whose equations are:

$$x = R \cdot arc(\varphi^*) = R \cdot \arcsin\left[\cos\varphi \cdot \sin(\lambda - \lambda_K)\right]$$

$$y = R \cdot arc(\lambda^*) = R \cdot \arcsin\left[\frac{\sin\varphi}{\sqrt{1 - \cos^2\varphi \cdot \sin^2(\lambda - \lambda_K)}}\right]$$

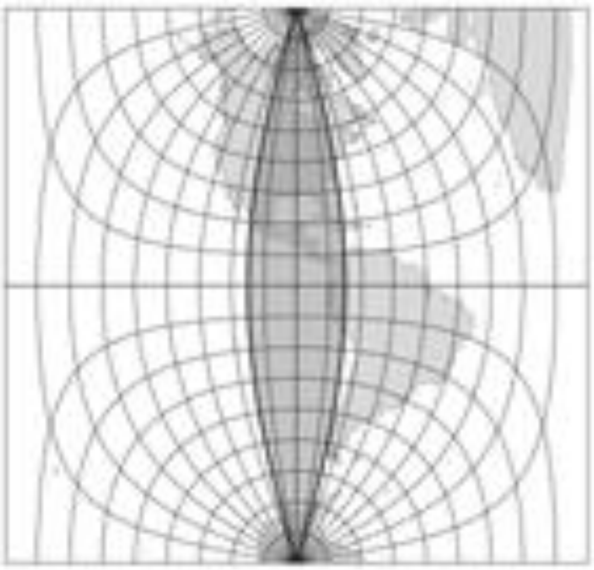
This is an *aphylactic* projection (neither conformal, nor equal-area), the central meridian (metaequator) and the metameridians are true scale.



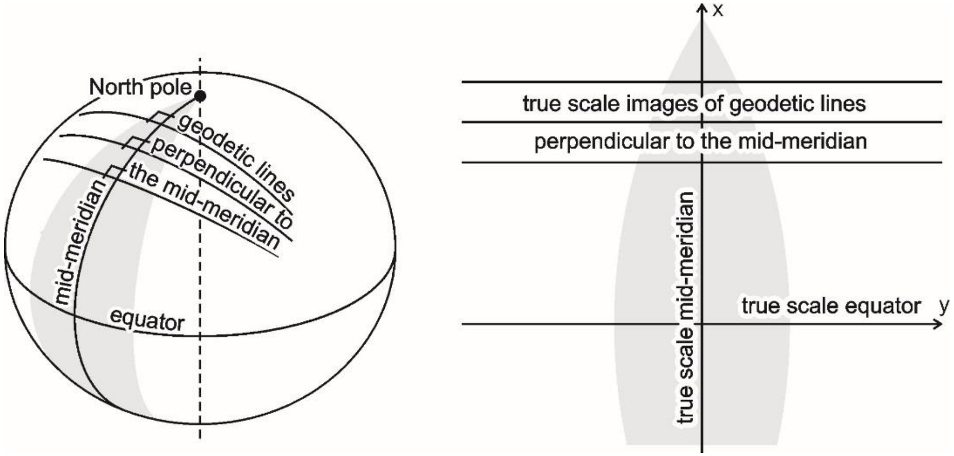
Transverse cylindrical projections of the sphere (3)

Application of the Cassini-projection for the traditional technology for production of globes:

the earth surface is partitioned into spherical lunes, bounded by two meridians with longitude differences of 30°, and these are mapped by Cassini projection onto zones (practically *paper gores*) which will be glued onto the naked globe.



The ellipsoidal generalization of the Cassini projection: an ellipsoidal lune will be mapped onto a planar zone.



Mapping an ellipsoidal lune onto a zone by the Cassini-Soldner projection

The image of the central meridian is a true scale straight line, and the images of the geodetic lines (orthodromes), perpendicular to it, including the equator, are true scale parallel straight lines: *Cassini-Soldner projection*. The coordinate axes *x* and *y* are the images of the central meridian and the equator. The coordinates *x* and *y* can be obtained from the direct and inverse principal geodetic problem related to the ellipsoid. Explicit projection equations are established by a power series expansion:

$$x = \int_0^\Phi M(\Phi)\, d\Phi + N(\Phi) \cdot \tan\Phi \cdot \left[\cos^2\Phi \cdot \frac{(\Lambda - \Lambda_0)^2}{2} + \left(5 - \tan^2\Phi + 6 \cdot \frac{e^2 \cdot \cos^2\Phi}{1 - e^2} \right) \cdot \cos^4\Phi \cdot \frac{(\Lambda - \Lambda_0)^4}{24} \right]$$

$$y = N(\Phi) \cdot \left[\cos\Phi \cdot (\Lambda - \Lambda_0) - \tan^2\Phi \cdot \cos^3\Phi \cdot \frac{(\Lambda - \Lambda_0)^3}{6} - \left(8 \cdot \frac{1 - e^2 \cdot \sin^2\Phi}{1 - e^2} - \tan^2\Phi \right) \cdot \tan^2\Phi \cdot \cos^5\Phi \cdot \frac{(\Lambda - \Lambda_0)^5}{120} \right]$$

The projection is aphylactic, the distortions in a narrow environment of the central meridian are small. It was used for large scale topographic maps in several countries during the XIX-XX century (e.g. France, Bavaria, the Habsburg Monarchy, Great-Britain, Russia, Israel).

Transverse conformal cylindrical projections of the sphere

In the conformal case, the projection equations are referred to the metacoordinates φ*, λ*:

$$x = R \cdot \cos\varphi_s * \cdot \ln\tan\left(45° + \frac{\varphi *}{2}\right) = \frac{R}{2} \cdot \cos\varphi_s * \cdot \ln\left(\frac{1+\sin\varphi *}{1-\sin\varphi *}\right)$$

$$y = R \cdot \cos\varphi_s * \cdot arc\,\lambda *$$

The transformation formulae by substituting 0° into φ_K:

$$\varphi* = \arcsin\left[\cos\varphi \cdot \sin(\lambda - \lambda_K)\right]$$

$$\lambda* = \arcsin\left[\frac{\sin\varphi}{\sqrt{1-\cos^2\varphi \cdot \sin^2(\lambda - \lambda_K)}}\right]$$

Thus the projection equations of the transverse *conformal* cylindrical (frequently „transverse Mercator") projection are:

$$x = R \cdot \cos\varphi*_s \cdot \ln\tan\left[45° + \frac{\arcsin(\cos\varphi \cdot \sin(\lambda - \lambda_K))}{2}\right] = \frac{R}{2} \cdot \cos\varphi*_s \cdot \ln\left(\frac{1+\cos\varphi \cdot \sin(\lambda - \lambda_K)}{1-\cos\varphi \cdot \sin(\lambda - \lambda_K)}\right)$$

$$y = R \cdot \cos\varphi*_s \cdot \arcsin\left[\frac{\sin\varphi}{\sqrt{1-\cos^2\varphi \cdot \sin^2(\lambda - \lambda_K)}}\right]$$

where $\pm\varphi*_s$ are the metalatitudes of the true scale metaparallels. Between them the linear and area scales are reduced, outside them they increase. The earth sphere is partitioned onto congruent spherical lunes with longitude difference of 30° and they are mapped onto plane. Printing them on paper, they are used for preparing paper „gores" in the traditional globe making.

„Transverse Mercator" projections of the ellipsoid (1)

The application of a double projection wouldn't be advantageous due to the greater distortions of the conformal ellipsoid-sphere transformation on the areas far from the true scale parallels, therefore it can not be applied.

The *ellipsoidal lune* will be mapped onto the plane by a complex function which maps the mid-meridian Λ_K of the lune to a *true scale straight line*. After the power series expansion of this complex function with respect to $(\Lambda - \Lambda_K)$ and separating it into a real and an imaginary part, the eqations of the *Gauss-Krüger* projection will be obtained:

$$X_{GK} = A_0 - A_2 \cdot arc^2(\Lambda - \Lambda_K) + A_4 \cdot arc^4(\Lambda - \Lambda_K) - \ldots$$

$$Y_{GK} = A_1 \cdot arc(\Lambda - \Lambda_K) - A_3 \cdot arc^3(\Lambda - \Lambda_K) + A_5 \cdot arc^5(\Lambda - \Lambda_K) - \ldots$$

where

$$A_0 = \int_0^{\Phi} M(\Phi)\, d\Phi$$

The length of the meridian arc between the equator and the latitude Φ can be calculated numerically. Moreover, the recursive formula for multiplicators A_i:

$$A_i = \frac{1}{i} \cdot \frac{N(\Phi) \cdot \cos\Phi}{M(\Phi)} \cdot \frac{dA_{i-1}}{d\Phi}$$

To describe these multiplicator functions A_i, the notation η^2 will be introduced:

$$\eta^2 = \frac{e^2}{1 - e^2} \cdot \cos^2\Phi = (e')^2 \cdot \cos^2\Phi$$

„Transverse Mercator” projections of the ellipsoid (2)

Then the multiplicators A_i of the powers $(\Lambda-\Lambda_K)$ are the following:

$$A_1 = N(\Phi)\cdot\cos\Phi$$

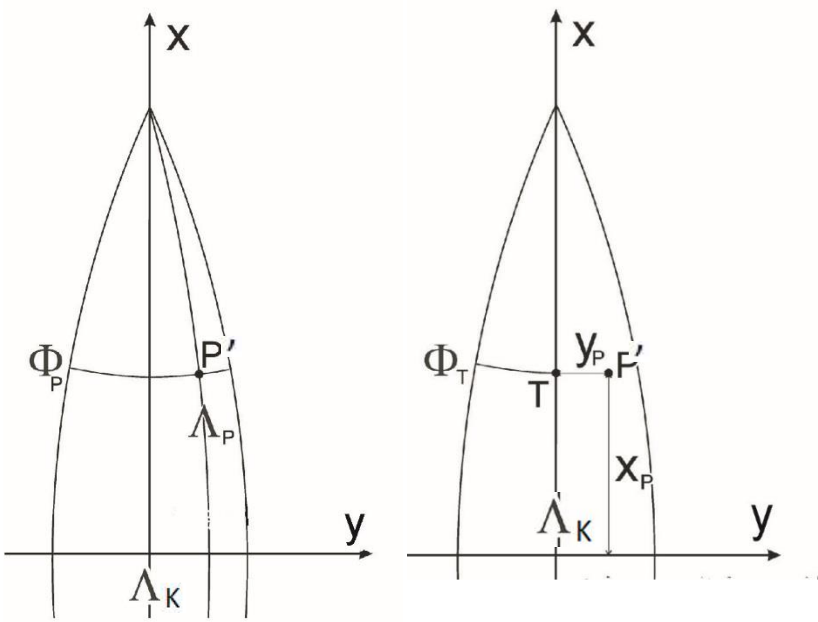
$$A_2 = -\frac{N(\Phi)}{2}\cdot\tan\Phi\cdot\cos^2\Phi$$

$$A_3 = -\frac{N(\Phi)}{6}\cdot\cos^3\Phi\cdot\left(1-tg^2\Phi+\eta^2\right)$$

$$A_4 = \frac{N(\Phi)}{24}\cdot\tan\Phi\cdot\cos^4\Phi\cdot\left(5-tg^2\Phi+9\cdot\eta^2+4\cdot\eta^4\right)$$

$$A_5 = \frac{N(\Phi)}{120}\cdot\cos^5\Phi\cdot\left(5-18\cdot tg^2\Phi+tg^4\Phi\right)$$

$$A_6 = -\frac{N(\Phi)}{720}\cdot\tan\Phi\cdot\cos^6\Phi\cdot\left(61-58\cdot tg^2\Phi+tg^4\Phi\right)$$



Conversely, the ellipsoidal geographic coordinates (Φ, Λ) can be obtained from the coordinates X_{GK}, Y_{GK} as follows. Let us consider the straight line, starting from the point P'(X_{GK}, Y_{GK}) and perpendicular to the mid-meridian Λ_K. Their intersection point is denoted by T. The latitude Φ_T of the T can be got from the solution of the nonlinear equation

$$X_{GK} = \int_0^{\Phi_T} M(\Phi)\, d\Phi$$

„Transverse Mercator" projections of the ellipsoid (3)

The geographic coordinates issue from the inverse equations

$$arc\Phi = arc\Phi_T - B_2 \cdot (Y_{GK})^2 + B_4 \cdot (Y_{GK})^4 - B_6 \cdot (Y_{GK})^6 - \ldots$$

$$arc\Lambda = B_1 \cdot (Y_{GK}) - B_3 \cdot (Y_{GK})^3 + B_5 \cdot (Y_{GK})^5 - \ldots$$

where

$$B_1 = \frac{1}{N(\Phi_T) \cdot \cos\Phi_T}$$

$$B_2 = \frac{(1+\eta_T{}^2)}{2 \cdot N^2(\Phi_T)} \cdot \tan\Phi_T$$

$$B_3 = \frac{(1 + 2 \cdot \tan^2\Phi_T + \eta_T{}^2)}{6 \cdot N^3(\Phi_T) \cdot \cos\Phi_T}$$

$$B_4 = \frac{(5 + 3 \cdot \tan^2\Phi_T + 10 \cdot \eta_T{}^2 - 4 \cdot \eta_T{}^4 - 9 \cdot (e')^2)}{24 \cdot N^4(\Phi_T)} \cdot (1 + \eta_T{}^2) \cdot \tan\Phi_T$$

$$B_5 = \frac{(5 - 2 \cdot \eta_T{}^2 + 28 \cdot \tan^2\Phi_T - 3 \cdot \eta_T{}^4 + 8 \cdot (e')^2 + 24 \cdot \tan^4\Phi)}{120 \cdot N^5(\Phi_T) \cdot \cos\Phi_T}$$

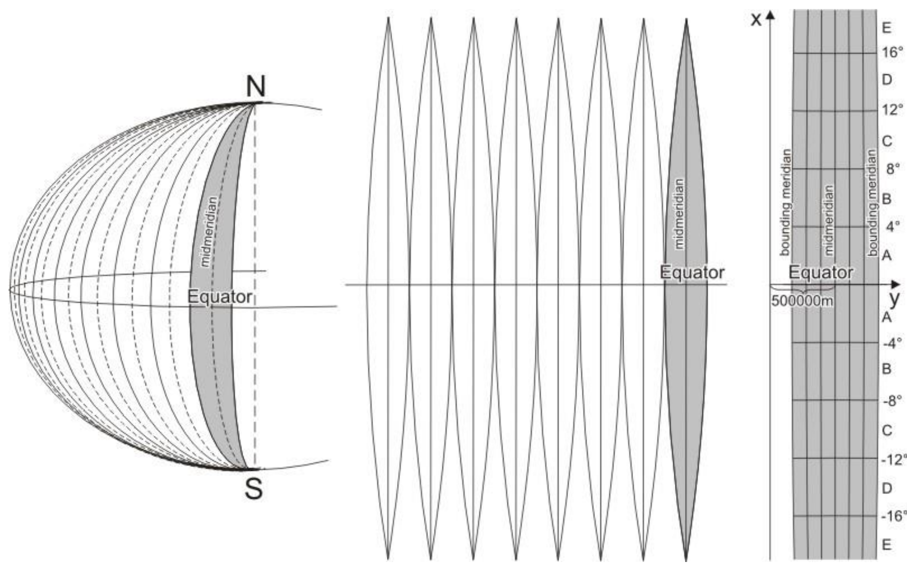
$$B_6 = \frac{(61 + 90 \cdot \tan^2\Phi_T + 298 \cdot \eta_T{}^2 + 45 \cdot \tan^4\Phi_T - 252 \cdot (e')^2 - 3 \cdot \eta_T{}^4)}{720 \cdot N^6(\Phi_T)} \cdot (1 + \eta_T{}^2) \cdot \tan\Phi_T$$

and $\eta_T{}^2 = \frac{e^2}{1-e^2} \cdot \cos^2\Phi_T = (e')^2 \cdot \cos^2\Phi_T$

Global map projections using the „transverse Mercator" projection of the ellipsoid (1)

Gauss-Krüger world map system

The Krasovskiy S42 ellipsoid is divided into 60 congruent parts – *ellipsoidal lunes with a longitude difference of 6°* – along meridians. All these figures are mapped *by the same Gauss-Krüger projection* to a *zone* so, that the images of the mid-meridians and the equator – straight lines – are the axes of symmetry of the zones. Their axis *y* coincides with the equator, and the axis *x* on the map is translated from the mid-meridian to west by 500000m (false easting), so that none of the coordinates *y* in the zone are negative. The zones are numbered starting from the Date Line eastwards from 1 to 60.



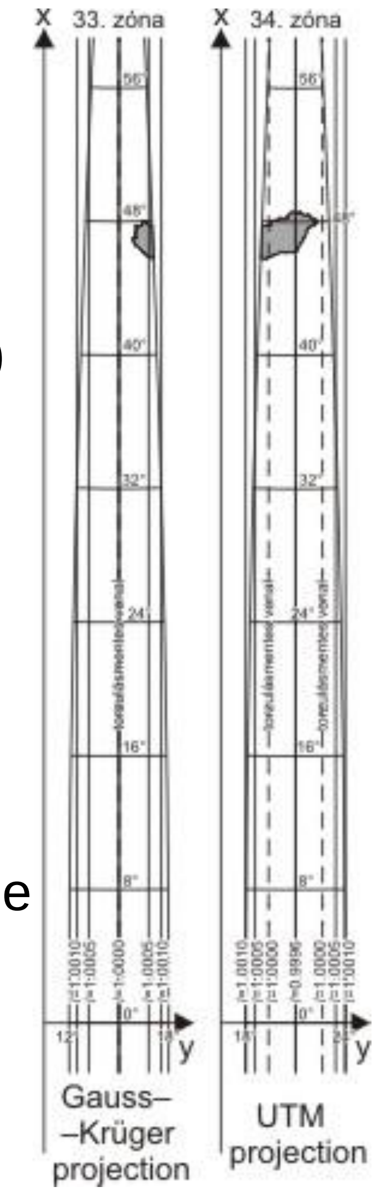
Global map projections using the „transverse Mercator” projection of the ellipsoid (2)

Every zones of 6° are partitioned by parallels into *geographic quadrangles with a latitude difference of 4°*. They are denoted by letters of the English alphabet using them from A to V, starting from the equator both to north (N) and to south (S). map sheet of 1:1 000 000 scale represents an area of 6°x4°.

The integer part of the coordinate number *x* (rounded to meter) consists generally of 7 digits. The same of the coordinate number *y* is composed of only 6 digits, and before them stands the second digit of the serial number of the zone („leading number”). It has not to be taken into consideration in the coordinate calculations, but it gives some information about the location of the point in question on the Earth.

Distortions:

Inside of a zone of 6°, the mid-meridian is free of distortions; the linear scales l are maximum at the intersection of the equator and the bounding meridians (l_{max}≈1.0013), therefore lunes of 2° (l_{max}≈1.00015) and 3° (l_{max}≈1.00034) are usual in some countries mostly on Bessel ellipsoid for geodesic purposes.



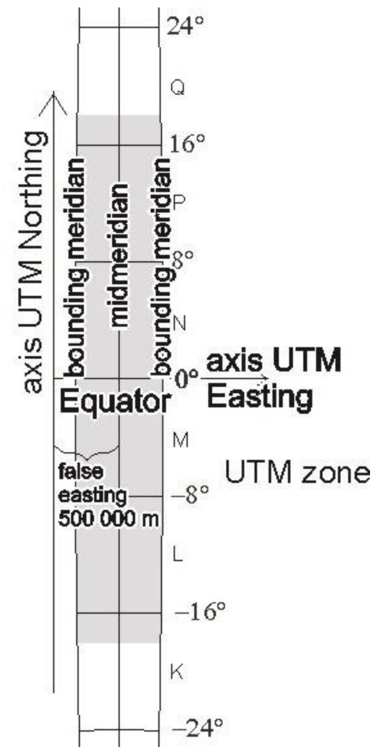
Global map projections using the „transverse Mercator" projection of the ellipsoid (3)

Universe Transverse Mercator (UTM) projection

The UTM projection is the *modification of the Gauss-Krüger* one with a reducing factor of 0.9996 in order to diminish the maximum scale and area distortions. Instead of x and y the UTM coordinates are denoted recently by „UTM Northing" and „UTM Easting" as usual. The false easting of 500000m results in positive y coordinates. On the southern hemisphere, a 10000000m false northing eliminates the negative x coordinates.

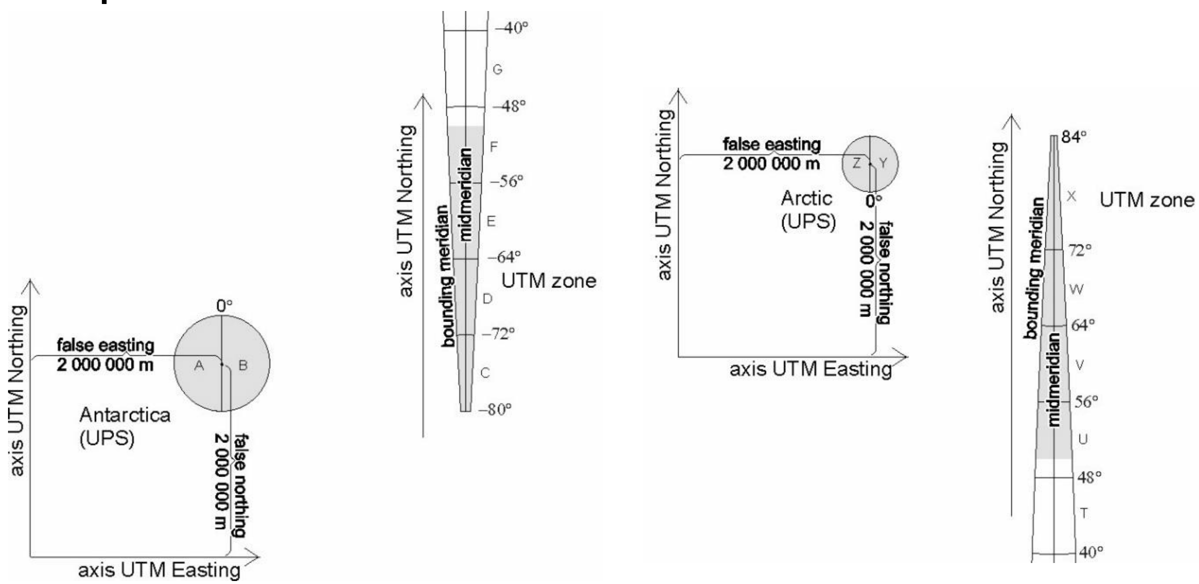
The UTM world map system

The represented territory is part of the reference ellipsoid WGS84 (earlier Hayford) between the 80th parallel south and the 84th parallel north. It is partitioned by meridians into congruent lunes whose longitude difference is 6°.
Every 60 lunes are partitioned by parallels into geographic quadrangles with a latitude difference of 8°, but the last one between 72° and 84° of north has 12° difference. They are denoted by letters of the English alphabet using them from C to X (except for I and O) going from south to north. Four sheets of 1:1 000 000 scale map represent an area of 6°x8°.



Global map projections using the „transverse Mercator” projection of the ellipsoid (4)

The Antarctic Region to the 80th parallel south is represented in UPS (Universe Polar Stereographic) projection, and its western and eastern half parts are denoted by A and B. The Arctic Region from the 84th parallel north is represented in UPS, too, and its eastern and western half parts are denoted by Y and Z.



Distortions of the UTM:

There are **two true scale straight lines** parallel with the mid-meridian. (Their intersections on the equator have ±1°37’15.2” longitude difference from the mid-meridian intersection.) The arc lengths and the areas decrease between the true scale straight lines, while they increase outside them. The linear scale along the mid-meridian equals 0.9996, and its maximum value (≈1.00097) is at the intersection of the equator and the bounding meridians.

Home work:

Given the WGS84 coordinates Φ=47°28’29.262”, Λ=19° 3’43.303”.

1. Asked the rough approximation of the UTM coordinates by the transverse conformal cylindrical projection, considering the Φ,Λ coordinates as spherical (R=6367449.1m, Λ_K=21°)
2. Considering the Φ,Λ coordinates as ellipsoidal, asked the terms of the UTM coordinates and their sums by gradual approximation, taking one, two and three terms.