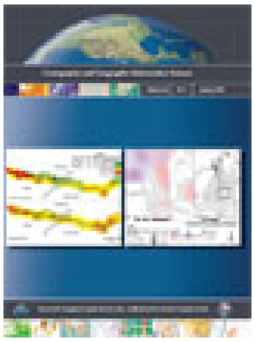


Cartography and Geographic Information Science

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tcag20

A statistical reinterpretation and assessment of criteria used for measuring map projection distortion

Krisztián Kerkovits

CARTOGRAPHY AND GEOGRAPHIC INFORMATION SCIENCE
https://doi.org/10.1080/15230406.2020.1768439

Taylor & Francis
Taylor & Francis Group

ARTICLE

A statistical reinterpretation and assessment of criteria used for measuring map projection distortion

Krisztián Kerkovits

Department of Cartography and Geoinformatics, ELTE Eötvös Loránd University, Budapest, Hungary

ABSTRACT
In map projections theory, various criteria have been proposed to evaluate the mean distortion of a map projection over a given area. Reports of studies are not comparable because researchers use different methods for estimating the deviation from the undistorted state. In this paper, statistical methods are extended to be used for averaging map projection distortions over an area. It turns out that the measure known as the Airy–Kavrayskiy criterion stands out as a simple statistical quantity making it a good candidate for standardization. The theoretical arguments are strengthened by a practical map projection optimization exercise.

ARTICLE HISTORY
Received 21 October 2019
Accepted 9 May 2020

KEYWORDS
Map projection; distortion value; logarithmic functions; statistics; Airy–Kavrayskiy criterion

1. The scope of this study

All map projections have some amount of distortion but every cartographer knows that certain mappings result in a less distorted map than others. This is especially true for regional maps, where the disadvantageous parts of map projections can be placed outside of the mapped area. This phenomenon encouraged the study of map projection distortion, and researchers needed to numerically express the quantity of distortion over a geographical region.

The concepts of map projection distorion were developed by Tissot (1878) who demonstrated that a map projection fulfilling certain differentiability conditions maps every infinitesimal circle on the reference frame to an infinitesimal ellipse. Another fundamental study of this field is due to Airy (1861). He also investigated map distortion on the infinitesimal scale while developing a minimum-distortion map projection. In addition, he introduced the concept of distortion value over an area, which is the cumulated effect of the local distortions observed in the infinitesimal scale. His formulae will be listed in section 4.1.

Airy's approach generated some controversy. First of all, it does not give enlargements and reductions the same weight. This leaded to various modifications of his original formula. The examples include Györffy (1990); Kavrayskiy (1934).

Furthermore, Peters (1979) argued that even Airy's and Tissot's basic concepts are not appropriate, referring to their formulae as "cartographically implausible". Following the approach of Peters (1975), several authors have studied map projection distortion using criteria measuring the distortion of finite distances, areas and shape developing diverse methods of the practical calculations (e.g. Canters, 2002; Gott et al., 2007). A recent study (Basaraner & Cetinkaya, 2019) investigated finite distortion not on random objects but on borders of countries and continents. The common in these studies is the assumption that Tissot's principles are inadequate to fully describe the finite distortions observed by map readers. However, Kerkovits (2019) recently demonstrated using various methods and arguments that this assumption is questionable. Finite distortions could be compared to corresponding infinitesimal measures with an unexpectedly high correlation coefficient (~0.99). For this reason, this study will not consider the distortion observed directly on finite objects as a completely separate phenomenon rather as mostly caused by infinitesimal distortions.

Another interesting approach, namely flexion and skewness defined by Goldberg and Gott (2007) to capture distortions of finite elements provided essentially different information on map projections (Kerkovits, 2019). Thus, flexion and skewness are superior to the method of finite objects to capture the non-affine properties of map projections. These measures will be discussed only marginally, readers should refer to Kerkovits (2018) for further information.

Given that a multitude of alternative map projection distortion criteria has been proposed, results obtained in different studies are difficult to compare. Furthermore some of them only consider angular distortion, others

CONTACT Krisztián Kerkovits kerkovits@map.elte.hu Pázmány Péter sétány 1/A, Budapest, Hungary, 1117

calculate areal distortion. The accumulated effect of these different kinds of distortion can only be calculated if the measures are on the same scale (i.e. they have the same dimension and weight). Laskowski (1997b) argued that a standard unit of distortion (SUD) needs to be defined to fit the results on the same scale. He suggested that the result of each calculation method should be rescaled, so that the distortion value of a well-chosen reference projection would be unit independently of the calculation method in question. Laskowski suggested more candidates as the reference projection but used only the Plate Carrée projection.

The central aim of this paper is to give a reinterpretation of current methods using mathematical statistics. This would enlighten which quantities are on the same scale. Furthermore, it can help the researchers in choosing the correct criteria based on strict mathematical considerations rather than subjective, experimental opinions.

For the sake of completeness, it must be noted that some researchers do not prefer mathematical calculation of map projection distortion at all. Baranyi (1968) was the first to develop new mappings using purely esthetical considerations. Recent studies often use the software *Flex Projector* (Jenny et al., 2008) to develop map projections on the basis of subjective opinions. While further studies are needed to investigate such mappings, it is clear that the assumption behind these developments is again that Tissot's theorem is not apt to capture the distortion perceived by the map reader. As long as Tissot's theorem can be considered as a sufficient method to quantify map distortions, the best map projection according to the mathematical methods should also have the best distortion pattern. To strengthen this statement, optimal world maps will be developed, and their properties will be investigated.

2. Distortion at the infinitesimal scale

2.1. Linear scale, areal scale and angular distortion

The definition of the *linear scale* l is:

$$l = \frac{\mathrm{d}t'}{\mathrm{d}t} \tag{1}$$

where t' is the distance measured on the map and t is the distance on the reference surface. Unless stated otherwise, the formulae in this paper do not take any assumption on the reference frame. It may be either a sphere, an ellipsoid of revolution, or even some more complicated surface.

The linear scale is *dimensionless*, as it is the derivative of a distance with respect to a distance. Furthermore, it is *multiplicative*. Kerkovits (2018) stated that applying two transformations one after the other (e.g. by using an intermediate sphere) the resulting linear scale will be the product of the linear scales calculated for the two mappings.

Distinction between multiplicative and additive behavior is important (Galton, 1879). Although the sum of multiplicative quantities may be defined, they are usually meaningless; and vice versa, the product of additive quantities should also be avoided for the same reason. Consequently, one should use the geometric mean to describe multiplicative data and the arithmetic mean for additive data.

The linear scale of a map projection depends not only on the location but also on the direction. To describe the linear scale at a point, it is usual to analyze the extremal values of it. In the following, a and b stand for the maximal and minimal linear scale at a given point.

Practical calculation of map distortion is based on Tissot's theorem, and formulae may be found in various textbooks. The reader may refer to Snyder (1987) for the formulae of a and b on the sphere and on the ellipsoid of revolution. Grafarend and Krumm (2006) list general formulae and derivation for any arbitrary reference frame.

The linear scale l may be expressed in terms of a and b for any direction:

$$l = \sqrt{a^2 \cos^2 \vartheta + b^2 \sin^2 \vartheta} \tag{2}$$

where ϑ denotes the angle on the reference frame measured from the direction of the maximal linear scale.

The definition of the areal scale p is:

$$p = \frac{\mathrm{d}S'}{\mathrm{d}S} = ab \tag{3}$$

where S denotes an area on the reference surface and S' stands for the area of its image. The dimension is area divided by area, meaning that it is *dimensionless*. It is also *multiplicative*, as the areal scale of a double projection is the product of the original areal scales.

Let one arm of the angle ϑ on the reference frame be the direction of the maximal linear scale a. Then the angular distortion i may be defined independently of the other arm of ϑ as:

$$i = \frac{\tan \vartheta'}{\tan \vartheta} = \frac{b}{a} \tag{4}$$

where ϑ' is the image of ϑ on the projection plane. Although recent studies usually calculate angular distortion using a different formula (i.e. the maximal angular deviation), this quantity has a strong connection:

$$2\omega = 2 \arcsin \frac{a-b}{a+b} = 2 \arcsin \left| \frac{i-1}{i+1} \right| \tag{5}$$

The author prefers to use b/a for angular distortion because angular distortion defined in this manner may be comparable to areal scale (Bayeva, 1987), as they are now measured on the same scale.

We can see from the formulae that this quantity and the areal scale are complementary to each other. The angular distortion is also *dimensionless* (angles have no dimension). Its multiplicative behavior can only be shown on orthogonal projections (i.e. if the images of parallels and meridians cross at right angles). Using other general projections, the accumulated angular distortion i of the two mappings is:

$$i = \frac{1}{4(i_1 i_2)^{3/2}} \left[2\left(\sqrt{i_1^5 i_2} + \sqrt{i_1 i_2^5} \right) \sin^2 \zeta + 2\left([i_1 i_2]^{5/2} + \sqrt{i_1 i_2} \right) \cos^2 \zeta - \sqrt{2}\left(i_1 i_2 \left[2\cos^4 \zeta + 2 i_2^4 \sin^4 \zeta + i_2^2 \sin^2(2\zeta) + 2 i_1^4 \left(\sin^2 \zeta + i_2^2 \cos^2 \zeta \right)^2 + i_1^2 \left(\sin^2[2\zeta] + i_2^4 \sin^2[2\zeta] + i_2^2 [\cos(4\zeta) - 5] \right) \right] \right)^{\frac{1}{2}} \right] \quad (6)$$

where i_1 and i_2 are the angular distortions of the original mappings and ζ is the angle on the intermediate surface between the directions of the maximal linear scales produced by the two projections. The formula above was obtained by a computer utilizing symbolic calculations using the definition of the angular distortion.

This formula means that the resulting angular distortion of a double projection is neither strictly multiplicative nor additive. However, note that $i = i_1 i_2$ if $\zeta = 0$, which suggests that this distortion may be considered multiplicative in a general sense.

2.2. The linear scale as a function of areal and angular distortions

In the previous section, three types of map projection distortions were expanded. A good map projection always tries to balance them. For example, even an equivalent projection should reduce angular distortion as much as possible. Laskowski (1997a) created his Tri-optimal projection by optimizing all three kinds of distortion simultaneously. But is it really needed to calculate all of them? Györffy (2016) revealed that the right side of equation (2) may be rearranged to include angular and areal distortions:

$$l = \sqrt{a^2 \cos^2 \vartheta + b^2 \sin^2 \vartheta} = \sqrt{\frac{p}{i} \cos^2 \vartheta + p i \sin^2 \vartheta} \quad (7)$$

It means that there is a strong functional relationship: the linear scale l depends only on the angular and areal distortions (i and p) and on the direction ϑ. That is, linear scale is fully dependent on the other two kinds of distortion. Furthermore, Kerkovits (2019) observed a strong ($\sim$0.99) *linear* correlation between the linear scale and the linear combination of areal and angular distortion. These two results imply that angular and areal distortions can totally describe the map distortions on the infinitesimal scale, it is not necessary to calculate the linear scale, as the other two quantities already include it implicitly.

3. Statistical quantities for a continuous deterministic population

In this paper, many concepts (like standard deviation, algebraic and geometric moments) will be borrowed from mathematical statistics. Although statistics are usually used to describe stochastic data, the definitions are also useful to describe populations of deterministic data, such as the distortions of map projections. The idea to use statistical quantities for the description of distortions originates from Gott et al. (2007).

Some generalization is needed to deal with the problem that the map projection distortion is continuous over the examined surface, it is a population of infinite members. However, all members of the population are known, since they are deterministic. In this paper, all statistical concepts are redefined by taking random elements from the population. As the number of the examined elements increases, the statistical measures of the sample converge to limiting values. A statistical measure for a deterministic population over the continuous interval Ω is defined to be this limiting value. E.g. the *expected value* or *algebraic mean* (denoted by angle brackets) will be the integral mean over the interval Ω:

$$\langle X \rangle = \frac{1}{\Omega} \int_{\Omega} X \, d\Omega \quad (8)$$

The *second algebraic moment* about a value c is defined here as:

$$\mu_2 = \sqrt{\langle (X - c)^2 \rangle} \quad (9)$$

This is the square root of the usual definition. The reason is to preserve dimensions: it is desired in this study that statistical quantities should have the same dimensions as the original values.

The well known formula of the *geometric mean* is reformulated here by taking the natural logarithm of both sides and using logarithmic identities:

$$\ln \mu_G = \langle \ln X \rangle \quad (10)$$

4. Measuring global distortion

The distortions l, p and i have a common property: all are 1 if and only if the corresponding distortion is not present at the examined point. If an area is considered, the question is: How much distortion appears on average in this area (Meshcheryakov, 1968)? Of course, one may also seek the extremal values. In this case, the result is well-defined but might be indeterminate due to the possible presence of infinite distortion at the edges of the map. On the other hand, the definition of the mean distortion over an area (in the following: *global distortion value*) is not defined consistently throughout the literature. It is clear that this quantity should average the *deviation* of the local quantities from 1. However, this deviation is usually evaluated in three ways:

4.1. "Subtract one and square" method

Airy (1861) defined the global distortion value E as (using the notation of this study):

$$E = \sqrt{\frac{1}{S}\int\int_S \frac{(p-1)^2 + (i^{-1}-1)^2}{2} \mathrm{d}S} \tag{11}$$

where S is the examined area on the reference frame, $p = ab$ is the areal scale and $i = b/a$ is the angular distortion.

Airy did not use it directly while seeking a minimum-distortion projection but minimized:

$$E = \sqrt{\frac{1}{S}\int\int_S \frac{(a-1)^2 + (b-1)^2}{2} \mathrm{d}S} \tag{12}$$

Both values use the same idea: the deviation from one is interpreted as the difference between the distortion and one. To eliminate negative numbers, squared differences are averaged. This can be viewed as either the quadratic mean of the deviations or as their second algebraic moment about 1. The biggest disadvantage is the different treatment of enlargements and reductions. It assumes, for example, that $p = 1/2$ is a smaller deviation from 1 than $p = 2$. Györffy (1990) corrected this by taking p^{-1} instead of p if $p < 1$.

4.2. Logarithmic functions

Logarithmic measures of map projections are attributed to Kavrayskiy (1934). It is also an attempt to correct the inconsistent behavior of Airy's global distortion value. Logarithms have the nice property: $\ln c = -\ln(1/c)$. Negative signs are still canceled by taking squares. Identities of logarithms make the definitions using p, i and a, b essentially equivalent:

$$E = \sqrt{\frac{1}{S}\int\int_S \frac{\ln^2 p + \ln^2 i}{2} \mathrm{d}S} = \sqrt{\frac{2}{S}\int\int_S \frac{\ln^2 a + \ln^2 b}{2} \mathrm{d}S} \tag{13}$$

This quantity will be referred to as the *Airy–Kavrayskiy criterion*.

The formula for the logarithm of the *geometric standard deviation* σ_G (Kirkwood, 1979) is quite similar:

$$\ln \sigma_G = \sqrt{\left\langle (\ln X - \langle \ln X \rangle)^2 \right\rangle} = \sqrt{\left\langle \ln^2 X \right\rangle - \langle \ln X \rangle^2} \tag{14}$$

There is only one substantial difference. Not the logarithm of the geometric mean is subtracted before taking the squares, but the logarithm of one (which is zero), indicating the geometric deviation from the state without distortion (note that $\ln^2 p = (\ln p - \ln 1)^2$). Therefore, the *second geometric moment of a population X about c* is defined here analogously to Kirkwood as:

$$\ln \mu_{2,G} = \sqrt{\left\langle (\ln X - \ln c)^2 \right\rangle} = \sqrt{\left\langle \ln^2 \frac{X}{c} \right\rangle} \tag{15}$$

Kavrayskiy's formula gives the *logarithm* of the second geometric moment about one. It is necessary, as geometric moments do not express that the dispersion is some value $\pm$ some deviation, rather it is some value $\times/$ some deviation (Limpert et al., 2001). It means, that 1 stands for no geometric deviation. This is not desired, the state without distortion should be 0. Therefore, the result is not exponentiated, as opposed to Kirkwood's formula.

Kavrayskiy's formula does not include the linear scale. The linear scale depends on the direction, so it must be averaged over all directions:

$$E = \sqrt{\frac{1}{S}\int\int_S \frac{1}{2\pi}\oint \ln^2 l \,\mathrm{d}\vartheta \,\mathrm{d}S} \tag{16}$$

where ϑ is the direction measured on the reference frame. This quantity will be referred to as the *Jordan–Kavrayskiy criterion* (Frančula, 1980).

4.3. Rational functions

Peters (1975) estimated the global distortion value by taking random line sections and addressed the inconsistency of Airy's formula by averaging

$$\frac{|t'-t|}{|t'+t|} = \left|\frac{\frac{t'}{t}-1}{\frac{t'}{t}+1}\right| \tag{17}$$

where t' is the planar distance and t is the distance on the reference frame. If the length of the line sections are reduced to an infinitesimal size and the number of the line sections approaches infinity, this average converges to

$$E = \frac{1}{S}\int\int_S \frac{1}{2\pi}\oint \left|\frac{l-1}{l+1}\right| d\vartheta\, dS \quad (18)$$

where ϑ is the direction, in which the linear scale l is measured.

This approach is not unique. Canters (2002) proposed a similar measure for the areal distortion. Behrmann (1910) optimized angular distortion i by minimizing the sum of

$$2\omega = 2\arcsin\frac{a-b}{a+b} = 2\arcsin\left|\frac{i-1}{i+1}\right| \quad (19)$$

There are two remarkable points of these approaches: This rational function associates enlargements and reductions of the same size to the same absolute values, but opposite signs. To eliminate negative signs, values are not squared, rather their absolute value is taken.

Laskowski (1997b) combined the different approaches and created myriads of possible global distortion values. He listed, for example, a "subtract one and take the absolute value" method and different variations by either taking the squares or the absolute values of rational and logarithmic functions. He also added many original ideas, which are not used in general.

5. Why to choose Kavrayskiy's criteria?

There is a continuous debate between researchers whether to use logarithmic or rational functions for the global distortion value. The following arguments will demonstrate that logarithmic functions are from many viewpoints more convenient than other methods.

5.1. Balance between areal and angular distortion

Bayeva (1987) showed that Airy's criteria do not give equal weight to areal and angular distortion. She stated that the distortion values for areal and angular distortions should be equal on equidistant projections. This ensures that areal and angular distortions are measured on the same scale, and their linear combination will be a meaningful quantity. She demonstrated that this requirement is fulfilled by the Airy–Kavrayskiy criterion: $\ln^2 p = \ln^2 i$ is true if either a or b is one. The rational function of Peters not considered by Bayeva is still a good candidate, as $|p-1|/(p+1) = |i-1|/(i+1)$ on all equidistant projections.

5.2. Meaningful operations

We should also investigate whether the requirements of dimension analysis are fulfilled. The functions proposed by Airy and Peters both calculate the deviation of a local scale x from 1 as $x-1$ (this is in the numerator of Peters' function). This operation is not recommended because linear and areal scales are multiplicative, and angular distortion can also show multiplicative behavior in certain cases. The comparison should rather be done by taking ratios, i. e. $x/1$.

The integrals in the criteria of Airy calculate the second *algebraic* moment, while the Airy–Kavrayskiy criterion is the logarithm of the second *geometric* moment. Geometric measures are preferred for multiplicative quantities because the product is preferred to the sum to describe their accumulated effect. Linear, areal, and angular distortions are dimensionless; therefore, the logarithm is an allowed operation on them. The Airy–Kavrayskiy criterion (the logarithm of the geometric moment) is also dimensionless but it is additive due to the properties of the logarithm: $\ln(xy) = \ln x + \ln y$. This additive behavior and the balanced scaling discussed in the previous section make it an allowed operation to take the weighted sum of angular and areal distortions (Bayeva, 1987), i.e. one may optimize a projection for

$$E = \sqrt{\frac{1}{S}\int\int_S q\ln^2 p + (1-q)\ln^2 i\, dS} \quad (20)$$

where $0 \geq q \geq 1$ expresses how areal distortion p is undesirable compared to angular distortion i. $q = 1/2$ results in good continent shapes, while $q = 2/3$ optimizes the linear scales (Kerkovits, 2019).

5.3. Invariance to scaling

Canters (2002) demonstrated that a similarity transformation applied to the map (i.e., a change in the nominal scale) results in a different global distortion value but the map-reader does not observe any change in the map projection. Canters suggested that the least possible distortion value should be considered.

A rescaling by c changes the areal scale p to pc^2. The minimal areal distortion value of Kavrayskiy's type is:

$$E' = \min_{c^2}\sqrt{\frac{1}{S}\int\int_S \ln^2(pc^2)\, dS} \quad (21)$$

To calculate the optimal c^2, E^2 will be minimized to get a simpler result. This is sufficient because E may never be negative. E^2 will be minimal only if its derivative with respect to c^2 is zero:

$$\frac{\partial E^2}{\partial c^2} = \frac{1}{S}\iint\limits_S \frac{2\ln(pc^2)}{c^2}\,\mathrm{d}S = \frac{2}{c^2 S}\left(\iint\limits_S \ln p\,\mathrm{d}S + \iint\limits_S \ln c^2\,\mathrm{d}S\right) = 0 \quad (22)$$

That is, the two integrals must be opposite numbers:

$$\iint\limits_S \ln p\,\mathrm{d}S = -\iint\limits_S \ln c^2\,\mathrm{d}S = -S\ln c^2 \quad (23)$$

Rearranged:

$$\ln c^{-2} = \frac{1}{S}\iint\limits_S \ln p\,\mathrm{d}S = \langle \ln p\rangle \quad (24)$$

I. e., c^{-2} is equal to the geometric mean of the areal scales. Substitution of c back to formula (21) yields:

$$E' = \sqrt{\frac{1}{S}\iint\limits_S (\ln p - \langle \ln p\rangle)^2 dS} = \sqrt{\left\langle(\ln p - \langle \ln p\rangle)^2\right\rangle} = \sqrt{\langle \ln^2 p\rangle - \langle \ln p\rangle^2} \quad (25)$$

This is exactly the logarithm of the geometric standard deviation. Using the same method, it turns out that the minimum of the Jordan – Kavrayskiy criterion with respect to scaling will also be the logarithmized geometric standard deviation of l.

It means, that scale-invariant measures are easily obtained using statistical methods. This phenomenon was first observed by Gott et al. (2007) on a finite sample of linear distortion. On the other hand, such an expressive solution is not found for the criteria of Airy and Peters.

5.4. Handling infinite distortions

One of the reasons why Peters (1975) advised the use of rational functions is that the distortions of map projections may be infinite in certain points. It is impossible to average a set of numbers if one element is infinite. The rational function of Peters is bounded, it may never be greater than one. Infinite distortions are only usual at the Poles, therefore, it is common to evaluate map distortions only within latitudes $\pm 85^\circ$ (Frančula, 1971; Gede, 2011).

It is obvious that it is infinitely bad if the linear scale is infinite. On the other hand, it is also infinitely bad if the linear scale is zero because the neighborhood of this point is collapsed into a single point. Airy's criteria qualify the first case being infinitely bad but in the latter case $(0-1)^2 = 1$. The rational function of Peters maps both cases to one, and logarithms associate these extremal cases to infinity.

Infinite distortion is rare in map projections, it appears at distinct points or lines. It is almost certain (it has one probability) that a random point chosen on the reference frame has finite distortion. Such populations still may have well-defined moments (mean, standard deviation, etc.) as improper integrals. Kerkovits (2020) showed that the Airy–Kavrayskiy criterion is Riemann-integrable over the full sphere for several map projections with pole-lines. Furthermore, the improper integral was convergent for two projections with infinite distortions along the Equator. On the other hand, one may show that the improper integral does not exist for Airy's original criteria over the full sphere using common projections with pole-line.

Using Kavrayskiy's criterion, there is no need to underestimate unbounded, large distortions typically present near the boundaries of world maps unlike the rational function of Peters. It is also not necessary to disregard polar regions beyond latitude 85°. Although it is possible to construct a projection, for which the Airy–Kavrayskiy criterion is not convergent, they are not used in practice.

5.5. No arbitrary units of measurement

Angular and areal distortion are dimensionless. The second moments were defined in this paper to have the same dimension, therefore, they are also dimensionless. The Airy–Kavrayskiy criterion (the logarithm of the dimensionless second geometric moment) is still dimensionless. This means that no unit of measurement must be defined to express its value. However, for practical purposes, one may define a unit of measurement for dimensionless quantities, like the degree for angles. In spite, the standard unit of distortion is unnecessary, if exclusively logarithmic criteria are used.

Laskowski (1997a) defined the distortion value of the Plate Carrée projection to be 1 SUD but he mentioned that the orthographic one may also be a good candidate. Kerkovits (2020) calculated using another possible definition of the Airy–Kavrayskiy criterion ($1/\sqrt{2}$ times the formulae listed in this study) that while the global distortion value of the spherical Plate Carrée projection is a complicated number, the distortion value of the spherical orthographic projection is exactly unit. Consequently, the Airy–Kavrayskiy criterion expresses the distortion value relative to this projection. This is a nice property because this is the mapping that depicts the reference frame as seen from a distant point.

6. Comparison for small distortions

It was a usual assumption regarding the criteria of Airy that (Frančula, 1971)

$$\frac{(i^{-1}-1)^2+(p-1)^2}{2} \approx (a-1)^2+(b-1)^2 \quad (26)$$

However, Györffy (1990) demonstrated that this connection is only true if the distortions are small (i.e. $l \approx 1$). This led to the idea of comparing the three methods (criteria of Airy, Kavrayskiy, and Peters) in the neighborhood of 1 using series expansion. It turned out that the criterion of Peters needed a multiplication by 4 to get comparable results:

$$(x-1)^2 = (x-1)^2 \quad (27)$$

$$\ln^2 x \approx (x-1)^2 - (x-1)^3 + \frac{11}{12}(x-1)^4 - \frac{5}{6}(x-1)^5 + \dots \quad (28)$$

$$4\left(\frac{x-1}{x+1}\right)^2 \approx (x-1)^2 - (x-1)^3 + \frac{3}{4}(x-1)^4 - \frac{1}{2}(x-1)^5 + \dots \quad (29)$$

This shows that if $x \approx 1$ all three criteria return nearly identical values. The rational function better approximates the logarithmic distortion value, as they share the same cubic term while Airy's original distortion value is only a second-degree truncation of the series. The same can be shown for distortion values using absolute values. The reader may refer to Figure 1, which visually strengthens the previous statements.

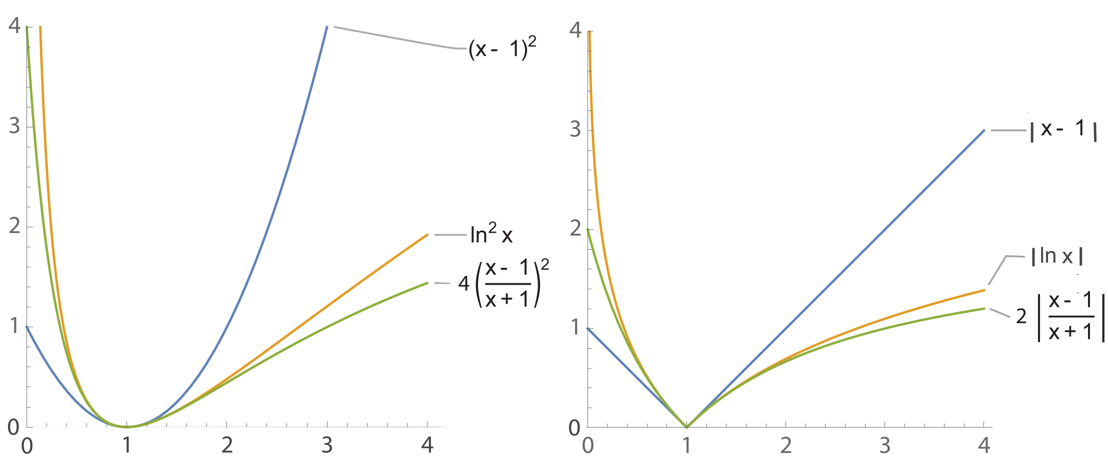
This means that these criteria can be used interchangeably provided that the distortion of the projection is small enough to disregard high degree terms of the series (e.g. on a cadastral map projection of a small country). Frančula (1971) also observed that logarithmic functions resulted in different conclusions only if the map distortion was high. The standard unit of distortion is still unnecessary because a simple multiplication could unify the scales of these distortion values for small distortions.

The original criterion of Airy is much simpler to use. The formula of the optimal azimuthal projection according to Airy's distortion value, for example, has been known for centuries (Airy, 1861) but the same problem is still unsolved for the Airy–Kavrayskiy criterion. Therefore, it is still a reasonable approach to approximate the global distortion value by Airy's approach.

However, Figure 1 also shows, that the squares and absolute values are not interchangeable even for small distortions.

7. To square or not to square?

There was a debate between Frančula and Peters in the journal Kartographische Nachrichten on the use of quadratic distortion values. Peters (1979) stated that the quadratic distortion values overestimate the extreme distortions usually present near the pole line and argued that optimal projections gained using this method are not visually appealing. Frančula (1980) answered that quadratic formulae were used by more researchers, and squaring is necessary to enjoy the advantages of the least squares method.



(a) Squared functions (b) Absolute value functions

Figure 1. Comparison of functions calculating the deviation from 1.

Canters (2002) repeated the arguments of Peters and favored absolute values to squares. He reasoned that the optimal standard parallel of the equidistant cylindrical projection according to Airy's criterion is 61.7°, which is unacceptable. On the other hand, Canters calculated the best standard parallel according to the an abslolute-valued criterion and yielded the more appealing value of 37.5°. It must be noted that Canters did not mention an important fact: It was not the only difference between the two methods that Airy's is quadratic and his is not, he also balanced the weights of enlargements and reductions by taking reciprocals like Györffy (1990). Furthermore, Grafarend and Niermann (1984) obtained a similar optimal value (42°) using the Airy–Kavrayskiy criterion, which is quadratic.

One could theoretically substitute $|\ln x|$ for all instances of $\ln^2 x$ in the formulae and expect that extreme distortions would influence the result in a more moderate way. Even the important $|\ln p| = |\ln i|$ identity will remain true for equidistant projections. The ideal standard parallel of the equidistant cylindrical projection becomes 30.36°. On the other hand, some arguments can be given against the use of absolute values:

- Least absolute values are usually favored to least squares methods if the presence of significant outliers usually emerging from measurement errors are expected. Map distortions are deterministic, measurement errors are theoretically impossible.
- Formula (13) does not hold for absolute values:

$$\frac{|\ln p| + |\ln i|}{2} = \max\{|\ln a|; |\ln b|\} \neq |\ln a| + |\ln b| \tag{30}$$

- For the original Airy–Kavrayskiy criterion, the best projection can be obtained using the Euler – Lagrange differential equation, and its exact solution is known for the problem of the best cylindrical projection for a spherical belt (Györffy, 1990; Kerkovits, 2017). However, the Euler – Lagrange differential equation yields infinitely many solutions for the best cylindrical projection when using the absolute values of logarithms. The reason is the max operator in formula ((30)), which makes the distortion value independent from the optimized $y(\varphi)$ function at the polar regions: practically any meaningful differentiable function can produce the same distortion value near the pole-lines.
- In section 5.3, it was revealed that the minimum areal distortion is reached when one rescales the projection with the geometrical mean of areal scales. Using absolute values, the optimal scaling number (calculated using the same method as in section 5.3.) depends on the median of the areal scales, which is way more cumbersome (though not impossible) to estimate for an infinite population.

For the previous reasons, it is not recommended to use the least absolute values method instead of the least squares method for map distortions.

8. Implementation notes

Kerkovits (2020) did some investigation on the practical calculation of the proposed Airy–Kavrayskiy criterion. Because the integral in the formula usually cannot be expressed in a closed form, a 2D numerical integration method is needed. Findings in the study cited show that most general-purpose integration rules produced acceptable results (~0.1% accuracy) and in general, the two-point Gaussian quadrature should be sufficient.

9. Optimizing map projection distortion using different distortion criteria

In the previous sections, it was demonstrated that the most meaningful quantity to represent the overall map distortion over an area is the Airy–Kavrayskiy criterion. To demonstrate the use of this criterion for map projection optimization and enable a comparison with results obtained with other criteria discussed in this paper, a world map projection was optimized according to four criteria:

(1) Airy's original: $\langle[(ab-1)^2 + (a/b-1)^2]/2\rangle$
(2) Airy's modified: $\langle[(a-1)^2 + (b-1)^2]/2\rangle$
(3) Rational function: $\langle([(ab-1)/(ab+1)]^2 + [(a/b-1)/(a/b+1)]^2)/2\rangle$
(4) Airy–Kavrayskiy: $\langle[\ln^2(ab) + \ln^2 a/b)]/2\rangle$

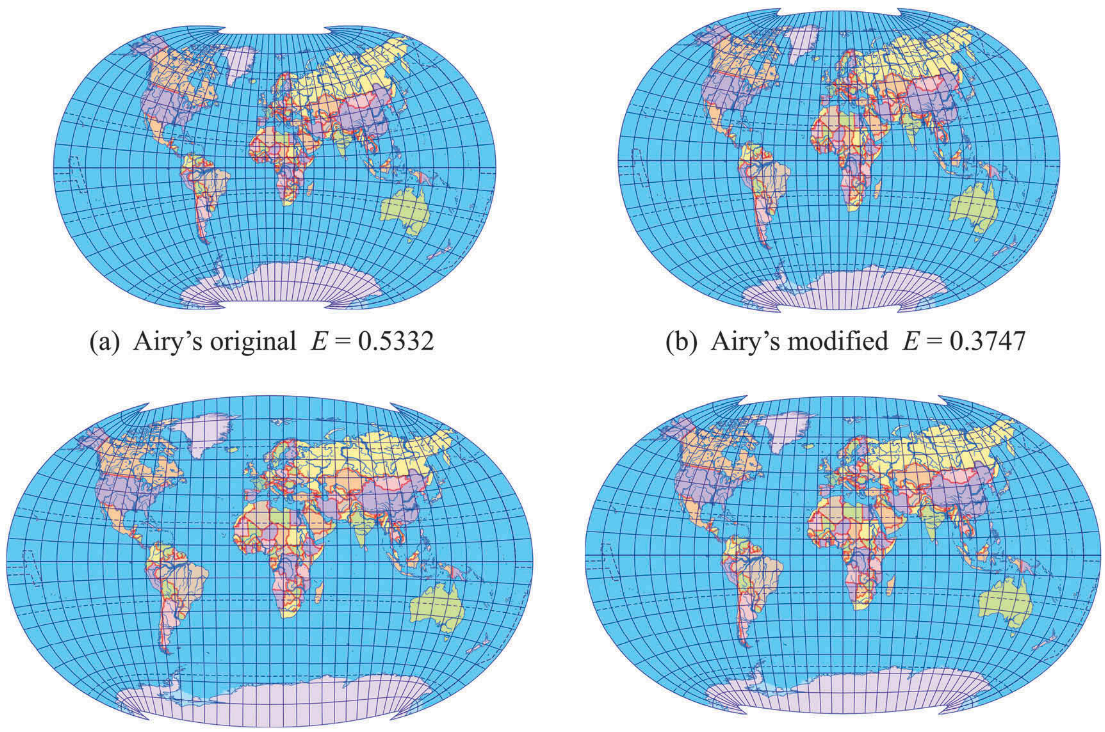
To minimize the distortion, coefficients of a fifth order polynomial were optimized by the Nelder–Mead method using the same polynomial as Canters (2002):

$$X = \xi_{01}\lambda + \xi_{03}\lambda^3 + \xi_{21}\varphi^2\lambda + \xi_{05}\lambda^5 + \xi_{23}\varphi^2\lambda^3 + \xi_{41}\varphi^4\lambda \tag{31}$$

$$Y = \upsilon_{10}\varphi + \upsilon_{30}\varphi^3 + \upsilon_{12}\varphi\lambda^2 + \upsilon_{50}\varphi^5 + \upsilon_{32}\varphi^3\lambda^3 + \upsilon_{14}\varphi\lambda^4 \tag{32}$$

where φ and λ are the spherical latitude and longitude, respectively.

The result is displayed on Figure 2. Several initial values were used to confirm that the results of the Nelder–Mead method are not just local minima. It was not possible to use the absolute value instead of quadratic expressions, as the optimizaton problem became



(a) Airy's original $E = 0.5332$

(b) Airy's modified $E = 0.3747$

(c) Rational function $E = 0.2151$

(d) Airy - Kavrayskiy $E = 0.3315$

Figure 2. The optimal map projection according to various criteria.

undetermined and Nelder–Mead method reported contradicting results for different initial values, hence no global minimum was found. The optimized area was the whole globe in the case of the logaithmic and rational measures, which can handle infinite distortion. However, calculation was bounded by latitude $\pm 85°$ while calculating Airy's criteria, otherwise the criterion would have diverged to infinity. Note that the reported distortion values are only for information, *they may not be compared to each other* because of the different method.

Although the maps are printed in the same nominal scale, the optimal maps according to Airy's criteria are smaller. This could be expected, as these criteria prefer reductions to enlargements. Furthermore, it is visible by naked eye that they did not give areal and angular distortions the same weight: the first map rather distorts areas but shapes of continents are more or less preserved. In the second map areal exaggeration is less visible but it has considerable angular distortion. Just as expected from the theoretical calculations, the rational and logarithmic measures balanced angular and areal distortion. Moreover, they yielded similar results, just as it was expected in section 6. The only visible difference is at the high latitudes where extreme distortion occurs. Here, the rational function did not give these extreme distortions enough weight and allowed to enlarge the area of Greenland a bit more than the logarithmic measure did.

At first glance, it is clear that even the optimal map according to Kavrayskiy's criterion cannot be used for serious purposes. Careful examination shows that the continents, in general, have good shapes, the overall map distortion is low. The only problem is the strange shape of the map frame. Györffy (2016) presented a mathematical way to obtain considerably better outline for the map while losing only 1% from the Airy–Kavrayskiy criterion. Therefore, it is possible to develop good-looking map projections using only objective, mathematical methods. This outline correction is not needed for regional maps: their outline is determined by the map frame. There were several successful studies to develop good-looking and low-distortion regional maps using only the Airy–Kavrayskiy criterion. (Györffy & Klinghammer, 2004; Kerkovits, 2019)

10. Some remarks on the differences between infinitesimal and finite distortion

A map projection is not an affine transformation. Nevertheless, the non-affine properties of the projections are not effectively captured by the popular method of comparing finite objects with their reprojected image. The recently introduced flexion and skewness seem to be a better alternative. Kerkovits (2018) examined these quantities, provided general formulae for the sphere and the ellipsoid of revolution, and demonstrated that these quantities *have dimension and are additive*. Unlike the linear and areal scales, they do need a unit of measurement; their dimension is 1/distance. This means that no unit of measurement can transform them to be comparable to traditional distortion values. It is not meaningful to ask whether a map projection has more skewness than angular distortion. The attempt of Goldberg and Gott (2007) to express flexion and skewness in SUD relative to the Plate Carrée projection lacked proper dimension analysis. Being additive quantities, it is not recommended to calculate the geometric moments of flexion and skewness, their algebraic moments are preferred. Note that it is impossible to calculate the logarithm of flexion and skewness because no transcendental functions are defined for quantities with dimensions.

11. Conclusions

To estimate the global distortion value of a projection, it is not necessary to use myriads of possible formulae as proposed by Laskowski (1997a). There are three kinds of map distortion (linear, areal and angular), from which the first one is dependent on the latter two. The logarithm of the second geometric moment about one (that is, the Airy–Kavrayskiy criterion) is a proper statistical quantity to represent the deviation of the map projection from an undistorted state. Angular and areal distortion values over an area are both dimensionless and they are by nature on the same scale if using the Airy–Kavrayskiy criterion. There is no need to define a standard unit of distortion to calculate a weighted average of angular and areal distortion.

The author of this study agrees that the best choice of a map projection heavily depends on the map theme. However, the distortion of map projections ought not to be a subjective quantity. We have exact, rigorous mathematical formulae to calculate linear and areal scales, which have been well known for centuries. Using definitions borrowed from mathematical statistics, the description of map distortion over an area should be well-defined. Utilizing the correct statistical measures and checking formulae according to dimensional analysis, the usage of Laskowski's SUD becomes obsolete.

It was demonstrated that the Airy–Kavrayskiy criterion is strongly connected to the definition of the standard deviation. If one seeks a low-distortion map projection according to this quantity one may be sure that not only the map distortion but also the variance of map distortion will be as low as possible within the constraints of the optimization. This means that one should not expect that extreme distortions will occur within large parts of the optimized area unless the optimization method does not allow better distortion distribution.

The map with the lowest possible distortion is not always the best map for all purposes. Other factors (like the shape of the map outline) are also important but such constraints can be considered mathematically during optimization either by constraining the coefficients or by solving differential equations (see e. g. Canters, 2002; Györffy & Klinghammer, 2004).

Disclosure statement

No potential conflict of interest was reported by the author.

ORCID

Krisztián Kerkovits http://orcid.org/0000-0001-7960-5579

Data availability statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

References

Airy, G. B. (1861). Explanation of a projection by balance of errors for maps applying to a very large extent of the earth's surface; and comparison of this projection with other projections. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, *22*(149), 409–421. https://doi.org/10.1080/14786446108643179

Baranyi, J. (1968). The problems of the representation of the globe on a plane with special reference to the preservation of the forms of continents. In S. Radó (Eds.), *Hungarian Cartographical Studies* (pp. 19–43). Foldmeresi Intezet.

Basaraner, M., & Cetinkaya, S. (2019). New measures for analysis and comparison of shape distortion in world map projections. *Cartography and Geographic Information Science*, *46*(6), 518–531. https://doi.org/10.1080/15230406.2019.1567394

Bayeva, Y. (1987). Kriterii otsenki dostoinstva kartografiches-kikh proyektsiy, ispolzuyemykh dlya sostavleniya kart mira. *Geodeziya i Aerofotosyomka*, (3), 109–112.

Behrmann, W. (1910). Die beste bekannte flächentreue Projektion der ganzen Erde. *Petermanns Mitteilungen*, *56*(3), 141–144.

Canters, F. (2002). *Small-scale map projection design*. Taylor & Francis.

Frančula, N. (1971). Die vorteilhaftesten Abbildungen in der Atlaskartographie (Doctoral dissertation). Reinischen Friedrich Wilhelms Universität, Hohen Landwirtschaftlichen Fakultät.

Frančula, N. (1980). Über die Verzerrungen in den kartographischen Abbildungen. *Kartographische Nachrichten*, *30* (6), 214–216.

Galton, F. (1879). The geometric mean, in vital and social statistics. *Proceedings of the Royal Society of London*, *29* (196–199), 365–367. https://doi.org/10.1098/rspl.1879.0060

Gede, M. (2011). Optimising the distortions of sinusoidal-elliptical composite projections. In A. Ruas (Ed.), *Advances in cartography and giscience* (Vol. 2, pp. 209–225). Springer.

Goldberg, D. M., & Gott, J. R. (2007). Flexion and skewness in map projections of the Earth. *Cartographica: The International Journal for Geographic Information and Geovisualization*, *42*(4), 297–318. https://doi.org/10.3138/carto.42.4.297

Gott, R. J., Mugnolo, C., & Colley, W. N. (2007). Map projections minimizing distance errors. *Cartographica: The International Journal for Geographic Information and Geovisualization*, *42*(3), 219–234. https://doi.org/10.3138/carto.42.3.219

Grafarend, E. W., & Krumm, F. W. (2006). *Map projections: Cartographic information systems*. Springer. https://doi.org/10.1007/978-3-540-36702-4

Grafarend, E. W., & Niermann, A. (1984). Beste echte Zylinderabbildungen. *Kartographische Nachrichten*, *34*(3), 103–107.

Györffy, J. (1990). Anmerkungen zur Frage der besten echten Zylinderabbildungen. *Kartographische Nachrichten*, *40*(4), 140–146.

Györffy, J. (2016). Some remarks on the question of pseudocylindrical projections with minimum distortions for world maps. In G. Gartner, M. Jobst, & H. Huang (Eds.), *Progress in cartography: Eurocarto 2015* (pp. 253–265). Springer. https://doi.org/10.1007/978-3-319-19602-2_16

Györffy, J., & Klinghammer, I. (2004). Die kleinmaßstäbige Darstellung von Europa in Projektionen geringster Verzerrung – Die Basis für einen Europa-Atlas. *Kartographische Nachrichten*, *54*(6), 262–268. https://doi.org/10.1007/BF03545455

Jenny, B., Patterson, T., & Hurni, L. (2008). Flex projector–interactive software for designing world map projections. *Cartographic Perspectives*, (59), 12–27. https://doi.org/10.14714/CP59.245

Kavrayskiy, V. V. (1934). *Matematicheskaya kartografiya*. Redbaza Goskartotresta.

Kerkovits, K. (2017). Legjobb valódi hengervetület nagy területek ábrázolására. *Geodézia és Kartográfia*, *69*(3), 29–32.

Kerkovits, K. (2018). Calculation and visualization of flexion and skewness. *Kartografija i Geoinformacije*, *17*(29), 32–45. https://doi.org/10.32909/kg.17.29.3

Kerkovits, K. (2019). Comparing finite and infinitesimal map distortion measures. *International Journal of Cartography*, *5*(1), 3–22. https://doi.org/10.1080/23729333.2018.1500255

Kerkovits, K. (2020). Quadrature rules to calculate distortions of map projections. *The Cartographic Journal*. Advance online publication. https://doi.org/10.1080/00087041.2020.1714278

Kirkwood, T. B. L. (1979). Biometrics. *Geometric Means and Measures of Dispersion*, *35* (4), 908–909. Retrieved from http://www.jstor.org/stable/2530139

Laskowski, P. (1997a). Part 2: Distortion-spectrum applications projection cross-breeding. *Cartographica: The International Journal for Geographic Information and Geovisualization*, *34*(3), 19–40. https://doi.org/10.3138/08N6-T847-U740-7458

Laskowski, P. (1997b). Part 1: Distortion-spectrum fundamentals a new tool for analyzing and visualizing map distortions. *Cartographica: The International Journal for Geographic Information and Geovisualization*, *34*(3), 3–18. https://doi.org/10.3138/Y51X-1590-PV21-136G

Limpert, E., Stahel, W. A., & Abbt, M. (2001). Log-normal distributions across the sciences: Keys and clues: On the charms of statistics, and how mechanical models resembling gambling machines offer a link to a handy way to characterize log-normal distributions, which can provide deeper insight into variability and probability—normal or log-normal: That is the question. *BioScience*, *51*(5), 341–352. https://doi.org/10.1641/0006-3568(2001)051[0341:LNDATS]2.0.CO;2

Meshcheryakov, G. A. (1968). *Teoreticheskie osnovy matematicheskoj kartografii*. Nedra.

Peters, A. B. (1975). Wie man unsere Weltkarten der Erde ähnlicher machen kann. *Kartographische Nachrichten*, *25* (5), 173–183.

Peters, A. B. (1979). Die Winkelsche Abbildung geringster Verzerrung: Anmerkung zum Gutachten von A. Heupel und J. Schoppmeyer. *Kartographische Nachrichten*, *29*(5), 179–182.

Snyder, J. P. (1987). *Map projections — A working manual*. Vol. 1395 US Government Printing Office. http://pubs.er.usgs.gov/publication/pp1395

Tissot, A. (1878). Mémoire sur la représentation des surfaces et les projections des cartes géographiques. *Nouvelles annales de mathématiques, journal des candidats aux écoles polytechnique et normale*, *2*(18), 385–397. http://www.numdam.org/item/NAM_1879_2_18__385_0/