Distortion factors

When mapping a sphere or ellipsoid to a plane, distortions always appear. It is impossible to map the whole curved surface without any angular, area or linear distortions.

Various types of projection distortions are expressed as quotients of mapped and original measures of infinitesimally small elements. The optimum value of these factors is one, as it means that the mapped and the original measures match.

Linear distortion factor (scale factor)

Let's draw a line between an arbitrary point A and B on the sphere (or ellipsoid), and denote the length of the line with s_{AB}, and the length of the projected figure of this line with $s_{A'B'}$. The quotient of these length, when point B is approaching to point A is called *linear distortion factor* or *scale factor* at point A:

$$l = \lim_{B \to A} \frac{s_{AB}}{s_{A'B'}}$$

The scale factor l usually depends on the *position* of the point and also on the *direction* of the line.

Angular distortion factor

Let's draw two lines on the sphere crossing each other at an arbitrary point A, and mark the angle of the crossing with γ. The projected figure of these lines is two arbitrary curves crossing each other at another γ' angle. The angular distortion at point A (denoted as i) is expressed as:

$$i = \frac{\tan \gamma'}{\tan \gamma}$$

The angular distortion factor i (just like the scale factor) usually depends on the *position* of the point and also on the *direction* of the lines.

Area distortion factor

Let's now draw a small circle around an arbitrary point A on the sphere, and mark its area with F. Let's mark the area of the depicted figure of this sphere on the projection plane with F'. The area distortion (denoted with τ) at point A is the quotient of these areas when the circle is infinitesimally shrunken to the point:

$$\tau = \lim_{F \to 0} \frac{F'}{F}$$

Tissot's theorem

(after Nicolas Auguste Tissot, French cartographer, 1824-1897)

Tissot's theorem introduces an approach to map projection distortions where these distortions are described by small ellipses throughout the map.

I.

Infinitesimally small unit circles on the Earth model are transformed into ellipses on the projection plane. These ellipses can be achieved from the circles by two perpendicular affine transformations. The ellipses are called *Tissot-indicatrices* or *ellipses of distortion*.

II.

At every point of the Earth model (either a sphere or an ellipsoid of revolution) there are two orthogonal directions which are perpendicular to one another on both the globe and the map. These directions are called principal directions.

The axes of the Tissot-indicatrix are always principal directions; the greater half-axis (denoted by a) is along the first (I) principal direction, while the smaller (b) is along the second (II) principal direction.

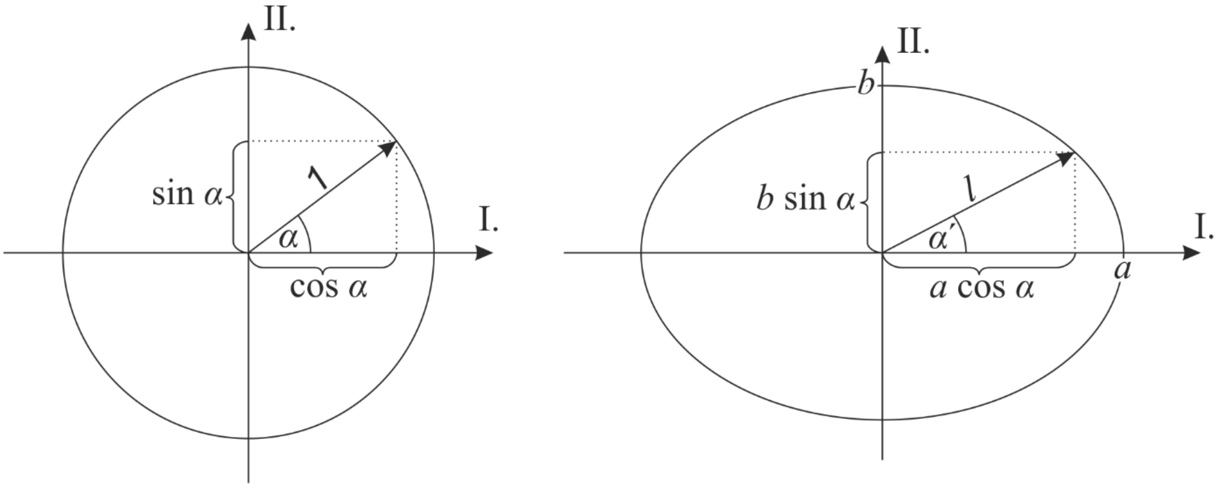

Figure 5. A unit circle on the sphere (left) and its projected image, a Tissot indicatrix (right).

The measures of the indicatrix, a and b can be used to express the various distortion factors of the projection. Draw a unit circle onto the sphere with a radial unit vector having and arbitrary α angle to the first principal direction, and observe its projected image, the indicatrix (Figure 5). The length of the projected unit vector equals l, the scale factor, and can be expressed based on the figure as:

$$l = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}$$

As we can see, the scale factor depends on the angle α, having a minimum value of b and a maximum of a.

The area distortion factor can be expressed as the quotient of the areas of the indicatrix and the unit circle:

$$\tau = \frac{\pi\, ab}{\pi} = ab$$

The angular distortion factor for angles that are measured from the first principal direction (such as α):

$$i = \frac{\tan \alpha'}{\tan \alpha} = \frac{\frac{b \sin \alpha}{a \cos \alpha}}{\frac{\sin \alpha}{\cos \alpha}} = \frac{b}{a}$$

III.

The difference of the original and the projected angle $(\alpha - \alpha')$ is called angular deformation. Its maximum value also can be derived from the measures of the distortion ellipse. First, let's express $\frac{\sin(\alpha-\alpha\prime)}{\sin(\alpha+\alpha\prime)}$ with a and b:

$$\frac{\sin(\alpha - \alpha')}{\sin(\alpha + \alpha')} = \frac{\sin \alpha \cos \alpha' - \cos \alpha \sin \alpha'}{\sin \alpha \cos \alpha' + \cos \alpha \sin \alpha'} = \frac{\frac{\sin \alpha \cos \alpha'}{\cos \alpha \sin \alpha'} - 1}{\frac{\sin \alpha \cos \alpha'}{\cos \alpha \sin \alpha'} + 1} = \frac{\frac{\tan \alpha}{\tan \alpha'} - 1}{\frac{\tan \alpha}{\tan \alpha'} + 1} = \frac{\frac{a}{b} - 1}{\frac{a}{b} + 1} = \frac{a - b}{a + b}$$

Therefore, $\sin(\alpha - \alpha')$ can be expressed as:

$$\sin(\alpha - \alpha') = \frac{a - b}{a + b} \sin(\alpha + \alpha')$$

As a and b are constants, and the maximum of $\sin(\alpha + \alpha')$ is 1 (when $\alpha + \alpha' = 90°$), so

$$\max(\sin(\alpha - \alpha')) = \frac{a - b}{a + b}$$

which leads us to

$$\max(\alpha - \alpha') = \arcsin \frac{a - b}{a + b}$$

as the maximum angular distortion value.