Pseudocylindrical projections

General properties

Pseudocylindrical projections in normal aspect share the following properties:

- Parallels (latitudes) are straight parallel lines
- The graticule has two axes of symmetry: the Equator and the central meridian

These properties imply that the projection formulas are:

$y = y(\varphi)$ which is an odd function [i.e. $y(-\varphi) = -y(\varphi)$]

$x = x(\varphi, \lambda)$ which is dd function of λ and even function of φ.

Distortions along the graticule lines

$\frac{\partial y}{\partial \lambda} = 0$, because y is independent of λ. It means that

$$h = \frac{\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2}}{\cos\varphi} = \frac{\frac{\partial x}{\partial \lambda}}{\cos\varphi}$$

$$\cot\Theta = \frac{\frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial \lambda} + \frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \lambda}}{\frac{\partial y}{\partial \varphi}\frac{\partial x}{\partial \lambda} - \frac{\partial x}{\partial \varphi}\frac{\partial y}{\partial \lambda}} = \frac{\frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \lambda}}{\frac{\partial y}{\partial \varphi}\frac{\partial x}{\partial \lambda}} = \frac{\frac{\partial x}{\partial \varphi}}{\frac{dy}{d\varphi}}$$

$$k = \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2} = \frac{dy}{d\varphi}\sqrt{\left(\frac{\frac{\partial x}{\partial \varphi}}{\frac{dy}{d\varphi}}\right)^2 + \left(\frac{\frac{\partial y}{\partial \varphi}}{\frac{dy}{d\varphi}}\right)^2} = \frac{dy}{d\varphi}\sqrt{\cot^2\Theta + 1} = \frac{dy}{d\varphi}\sqrt{\frac{\cos^2\Theta + \sin^2\Theta}{\sin^2\Theta}}$$
$$= \frac{dy}{d\varphi}\frac{1}{\sin\Theta}$$

Therefore $\frac{dy}{d\varphi} = k\sin\Theta$, so

$$\tau = hk\sin\Theta = \frac{\frac{\partial x}{\partial \lambda}}{\cos\varphi}\frac{dy}{d\varphi} = \frac{\partial x}{\partial \lambda}\frac{dy}{d\varphi}\frac{1}{\cos\varphi}$$

If a pseudocylindrical projection is equal-area, $\tau = \frac{\partial x}{\partial \lambda}\frac{dy}{d\varphi}\frac{1}{\cos\varphi} = 1$, therefore $\frac{\partial x}{\partial \lambda} = \frac{\cos\varphi}{\frac{dy}{d\varphi}}$, independent of λ, which means that x is a linear function of λ, therefore the spacing of the meridians (longitudes) is uniform.

Sinusoidal projections

Sinusoidal (Mercator-Sanson or Sanson-Flamsteed) projection

Let's construct a pseudocylindrical projection with true scale parallels and central meridian!

True scale parallels means: $h = \frac{\frac{\partial x}{\partial \lambda}}{\cos\varphi} = 1$, so $\frac{\partial x}{\partial \lambda} = \cos\varphi$. Therefore: $x = \lambda\cos\varphi$ (there could be an additional constant, but it must be zero because of the symmetry to the central meridian)

As the central meridian is perpendicular to the latitude lines, $\Theta = 90°$ along the central meridians, therefore $\sin\Theta = 1$. So $k = \frac{dy}{d\varphi}\frac{1}{\sin\Theta} = \frac{dy}{d\varphi} = 1$ here, which means that $y = \phi$. So the formulas of this projection in normal aspect:

$$x = \lambda \cos\varphi$$

$$y = \phi$$

Additionally, the area distortion factor: $\tau = hk\sin\Theta = 1\frac{dy}{d\varphi}\frac{1}{\sin\Theta}\sin\Theta = 1$, so this projection is equal-area.

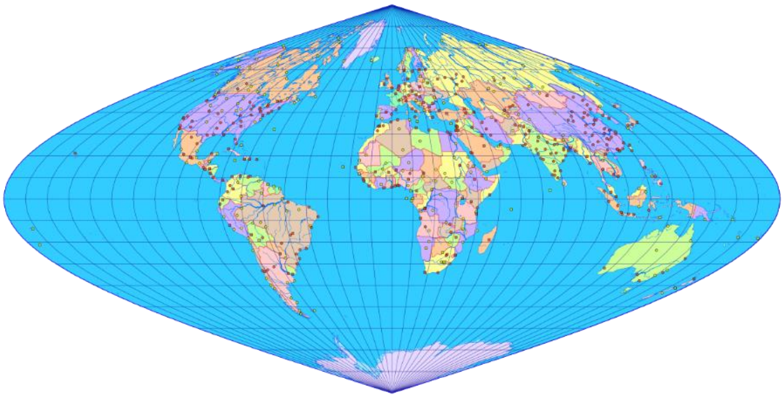

Figure 12. The world in Sinusoidal (Mercator–Sanson–Flamsteed) projection in normal aspect

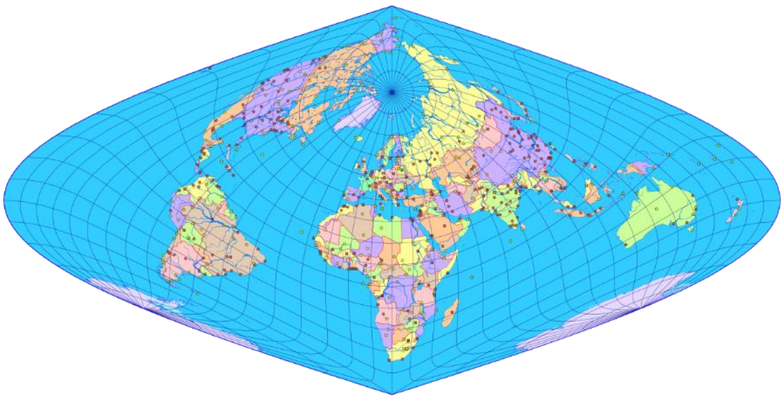

Figure 13. The world in Sinusoidal (Mercator–Sanson–Flamsteed) projection in oblique aspect

The **advantages** of this projection are its **equal area** nature and that **all parallels and the central meridian are true scale**. Its disadvantages are the great angular and scale distortions farther from the Equator and the central meridian.

It is rarely used for depicting the whole Earth, more often for regions along the Equator. This projection is also the base of several other projections.

Graticule renumbering transformations

A simple way of creating new projections from existing ones is the use of *graticule renumbering transformations*. These transformation usually “magnify” a smaller part of the projection graticule and fit the whole Earth in. To define such a transformation, we need to specify two functions: $\psi(\varphi)$ and $\zeta(\lambda)$ and the constants c and d.

If the original projection equations were:

$$x = x(\varphi, \lambda)$$

$$y = y(\varphi, \lambda)$$

then the transformed projection formulas are the following:

$$x = c \cdot x[\psi(\varphi), \zeta(\lambda)]$$

$$y = d \cdot y[\psi(\varphi), \zeta(\lambda)]$$

Functions $\psi(\varphi)$ and $\zeta(\lambda)$ should be strictly monotonically increasing and continuously differentiable and obviously $\psi(0) = 0$ and $\zeta(0) = 0$ (The Equator and the central meridian remains the same).

Wagner's transformation

Wagner's transformation (developed by Karlheinz Wagner) is a graticule renumbering transformation that preserves the uniform spacing of meridians and the equal-area nature of the base projection (i. e. if the base projection was equal-area and its meridians were equally spaced, the transformed projection inherits these properties).

Let's suppose that we want to use part of the original projection graticule between the latitudes $-\varphi_B$ and φ_B, and the longitudes $-\lambda_B$ and λ_B, which means that in the new projection these latitudes and longitudes become the pole lines and the boundary ($\pm 180°$) meridians.

As the transformation preserves the uniform spacing of meridians, it implies that $\zeta(\lambda)$ is a linear function:

$\zeta(\lambda) = n\lambda$, where $n = \frac{\lambda_B}{180°}$ or $n = \frac{\lambda_B}{\pi}$ (if calculating angles in radians).

The area of a spherical belt between the Equator and a given φ latitude on the unit sphere:

$$A(\varphi) = 2\pi(\sin\varphi - \sin 0) = 2\pi \sin\varphi$$

To preserve the equal-area nature of the projection during the transformation requires the following:

$$\frac{A(\varphi)}{A(90°)} = \frac{A(\psi(\varphi))}{A(\psi(90°))} = \frac{A(\psi(\varphi))}{A(\varphi_B)}$$

that is

$$\frac{2\pi \sin\varphi}{2\pi \sin 90°} = \frac{2\pi \sin\psi(\varphi)}{2\pi \sin\varphi_B}$$

after simplification

$$\sin\varphi = \frac{\sin\psi(\varphi)}{\sin\varphi_B}$$

so finally, using the $m = \sin\varphi_B$ substitution:

$$\psi(\varphi) = \arcsin(\sin\varphi_B \sin\varphi) = \arcsin(m \sin\varphi)$$

or

$$\sin\psi(\varphi) = m \sin\varphi$$

We still need to calculate the constants c and d. The transformation shrinks the geographic quadrangle (0 – 90°, 0 – 180°) to the one (0 – φ_B, 0 – λ_B), having areas $A_{original} = \pi \sin 90° = \pi$ and $A_{transformed} = n\pi \sin \varphi_B = \pi nm$. Constants c and d are the magnifying factors responsible for compensating this area reduction, therefore $cd = \frac{1}{mn}$. Let's make the two constants equal, so

$$c = d = \frac{1}{\sqrt{mn}}$$

Applying Wagner's transformation to the Sinusoidal projection

The original projection formulas of Sinusoidal projection are:

$$x = \lambda \cos \varphi$$

$$y = \phi$$

Therefore the transformed formulas:

$$x = c \cdot \zeta(\lambda) \cos \psi(\varphi) = \frac{1}{\sqrt{mn}} n\lambda \sqrt{1 - \sin^2 \psi(\varphi)} = \sqrt{\frac{n}{m}} \lambda \sqrt{1 - m^2 \sin^2 \varphi}$$

$$y = c \cdot \psi(\varphi) = \frac{1}{\sqrt{mn}} \arcsin(m \sin \varphi)$$

Instead of the parameters *m* and *n*, it is advisable to us other, more illustrative parameters. These are:

The length of the central meridian in proportion of the length of the Equator:

$$p = \frac{y(90°)}{x(0, 180°)} = \frac{\frac{1}{\sqrt{mn}} \arcsin(m \sin 90°)}{\sqrt{\frac{n}{m}} \pi} = \frac{\arcsin m}{n\pi}$$

The length of the pole line in proportion of the length of the Equator:

$$q = \frac{x(90°, 180°)}{x(0°, 180°)} = \frac{\sqrt{\frac{n}{m}} \pi \sqrt{1 - m^2}}{\sqrt{\frac{n}{m}} \pi} = \sqrt{1 - m^2}$$

We can calculate *m* and *n* from *p* and *q*:

$$m = \sqrt{1 - q^2}$$

$$n = \frac{\arcsin m}{p\pi}$$

Example: $p = \frac{1}{2}$ and $q = \frac{1}{2}$

It means that $m = \frac{\sqrt{3}}{2}$ and $n = \frac{2}{3}$, so $\frac{1}{\sqrt{mn}} = \frac{1}{\sqrt{\frac{1}{\sqrt{3}}}} = \sqrt[4]{3}$. Therefore the projection formulas become:

$$x = \sqrt[4]{3} \frac{2}{3} \lambda \sqrt{1 - \frac{3}{4} \sin^2 \varphi}$$

$$y = \sqrt[4]{3} \arcsin\left(\frac{\sqrt{3}}{2} \sin\varphi\right)$$

This equal-area projection was developed by Wagner and independently by Kavrayskiy, so it is either called Wagner I or Kavrayskiy VI projection, and widely used in the post-Soviet countries for world maps.

Blended projections

It is possible to create a new projection by averaging to other projections. This solution is called projection blending.

Let's use the general formulas $x_1 = x_1(\varphi, \lambda)$, $y_1 = y_1(\varphi, \lambda)$ and $x_2 = x_2(\varphi, \lambda)$, $y_2 = y_2(\varphi, \lambda)$ for the two base projections. The formulas of the blended projection become:

$$x = \frac{x_1(\varphi, \lambda) + x_2(\varphi, \lambda)}{2}$$

$$y = \frac{y_1(\varphi, \lambda) + y_2(\varphi, \lambda)}{2}$$

Example: blending the *Plate Carrée* (Equirectangular) and the *Sinusoidal* projection

Plate Carrée:

$$x_1 = \lambda$$
$$y_1 = \varphi$$

Sinusoidal:

$$x_1 = \lambda \cos\varphi$$
$$y_1 = \varphi$$

Blended:

$$x = \lambda \frac{1 + \cos\varphi}{2}$$
$$y = \varphi$$

As the Sinusoidal is an equal-area projection but the Plate Carrée is not, the resulting projection is also not equal-area. The length of the pole line is the half of the Equator.

Let's multiply these formulas with such a *c* constant that will make the area of the full projection grid equal to the unit sphere surface ($A_{sphere} = 4\pi$).

The value of square of *c* can be calculated by dividing the sphere surface by the area of the original projection grid ($A_{original}$). This latter is the area between the boundary ($\pm 180°$) meridians which can be produced by integrating the $x(\varphi, 180°)$ function between the two poles:

$$A_{original} = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi \frac{1 + \cos\varphi}{2} d\varphi = \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 + \cos\varphi \, d\varphi = \pi[\varphi + \sin\varphi]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi\left(\frac{\pi}{2} + 1 + \frac{\pi}{2} + 1\right)$$
$$= \pi(\pi + 2)$$

Therefore $c^2 = \frac{4\pi}{\pi(\pi+2)}$, so $c = \frac{2}{\sqrt{\pi+2}}$, resulting the projection equations:

$$x = \frac{2}{\sqrt{\pi + 2}} \lambda \frac{1 + \cos\varphi}{2}$$

$$y = \frac{2}{\sqrt{\pi + 2}} \varphi$$

This is the **Eckert V** projection: a sinusoidal pseudocylindrical projection with a pole line half the length of the Equator, and with equally spaced parallels and meridians. It is used mostly in Germany for world maps.

(developed by Friedrich Eduard Max Eckert-Greifendorff, 1868–1938, German cartographer)

Latitude renumbering

Another way of creating a new projection from an existing one is the method of *latitude renumbering*. It is a special case of graticule renumbering transformations, when only φ is substituted by $\psi(\varphi)$, and $\psi(0) = 0$ and $\psi(\pm 90°) = \pm 90°$, so the Equator and the poles remain at their original place, only the other latitudes are moved.

Latitude renumbering is used to transform a non-equal-area pseudocylindrical projection to an equal area one while keeping the outline of the world map. This requires the original projection to have the full projection grid area equal to the surface of the sphere and meridians equally spaced.

Transforming Eckert V into an equal-area projection using latitude renumbering

Let's substitute φ by $\psi(\varphi)$ in the projection formulas of Eckert V:

$$x = c\lambda \frac{1 + \cos\psi(\varphi)}{2}$$

$$y = c\psi(\varphi)$$

where $c = \frac{2}{\sqrt{\pi + 2}}$.

We define the $\psi(\varphi)$ function by requiring the surface of the spherical belt between 0° and φ to be equal to the area of the shape lying between the Equator and the ψ parallel in the projection. This latter can be gained by integrating the *x* formula when $\lambda = \pi$ (the boundary meridian, 180° in degrees).

$$A_{spherical} = 2\pi \sin\varphi$$

$$A_{projection} = 2c^2 \int_0^{\psi} \frac{\pi}{2}(1 + \cos\psi)d\psi = c^2\pi \int_0^{\psi} (1 + \cos\psi)d\psi = c^2\pi(\psi + \sin\psi)$$

therefore

$$2\pi \sin\varphi = c^2\pi(\psi + \sin\psi)$$

simplified:

$$2\sin\varphi = c^2(\psi + \sin\psi)$$

This is an implicit function (as $\psi(\varphi)$ cannot be expressed explicitly). The value of ψ for any given φ can be calculated by solving the above equation using some numerical method such as *secant method* (https://en.wikipedia.org/wiki/Secant_method).

Applying the above transformation on Eckert V produces the equal-area **Eckert VI** projection. To confirm that it is really equal-area, let's calculate the distortions along the graticule lines and the area distortion factor. For this we need to express the $\frac{d\psi}{d\varphi}$ derivative first by deriving the implicit function of $\psi(\varphi)$:

$$\frac{d}{d\varphi}(2\sin\varphi) = \frac{d}{d\varphi}[c^2(\psi + \sin\psi)]$$

$$2\cos\varphi = c^2\left(\frac{d\psi}{d\varphi} + \frac{d\psi}{d\varphi}\cos\psi\right)$$

$$\frac{d\psi}{d\varphi} = \frac{2\cos\varphi}{c^2(1 + \cos\psi)}$$

Now the distortions:

$$h = \frac{\partial x}{\partial \lambda}\frac{1}{\cos\varphi} = c\frac{1+\cos\psi}{2}\frac{1}{\cos\varphi}$$

$$k = \frac{dy}{d\varphi}\frac{1}{\sin\theta} = c\frac{d\psi}{d\varphi}\frac{1}{\sin\theta} = c\frac{2\cos\varphi}{c^2(1+\cos\psi)}\frac{1}{\sin\theta} = \frac{2\cos\varphi}{c(1+\cos\psi)}\frac{1}{\sin\theta}$$

$$\tau = hk\sin\theta = \ c\frac{1+\cos\psi}{2}\frac{1}{\cos\varphi}\frac{2\cos\varphi}{c(1+\cos\psi)}\frac{1}{\sin\theta}\sin\theta = 1$$

The τ area distortion factor equals 1, which means that this projection is really equal-area.

The Eckert VI projection is widely used for world maps especially in Germany.

Pseudocylindrical projections with elliptical meridians

The meridians in this group of projections are ellipse arcs.

Apian II projection

The Apian II projection projects the hemisphere into a circle. The central meridian is true scale. The latitude lines are parallel straight lines, while the meridians are ellipse arcs and their spacing is uniform.

The true scale central meridian implies, that

$$y = \varphi$$

The hemisphere is mapped into a circle which means that the $\lambda = \pm 90°$ meridians form a circle (see Figure 14). Therefore

$$x_{\lambda=90°} = \sqrt{\left(\frac{\pi}{2}\right)^2 - \varphi^2} = \frac{\pi}{2}\sqrt{1-\frac{4}{\pi^2}\varphi^2}$$

The uniform spacing of the meridians means that *x* is a linear function of λ, so

$$x = \frac{\lambda}{\frac{\pi}{2}}x_{\lambda=90°} = \lambda\sqrt{1-\frac{4}{\pi^2}\varphi^2}$$

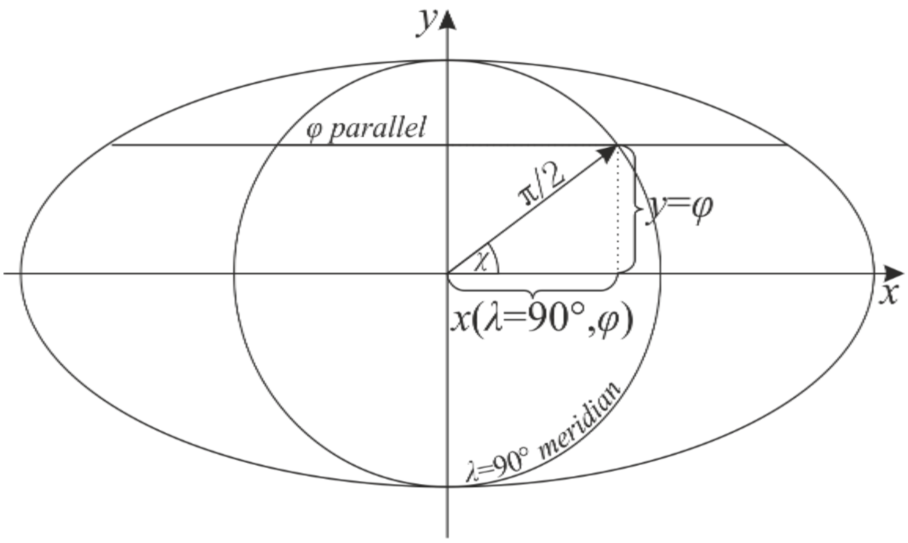

Figure 14. The $\lambda = \pm 90°$ meridians form a circle in Apian II projection.

So the projection formulas are:

$$x = \lambda \sqrt{1 - \frac{4}{\pi^2}\varphi^2}$$

$$y = \varphi$$

The *h* and *k* scale factors along the graticule lines are:

$$h = \frac{\frac{\partial x}{\partial \lambda}}{\cos\varphi} = \frac{\sqrt{1 - \frac{4}{\pi^2}\varphi^2}}{\cos\varphi}$$

$$k = \frac{dy}{d\varphi}\frac{1}{\sin\Theta} = \frac{1}{\sin\Theta}$$

Therefore the area distortion factor:

$$\tau = hk\sin\theta = \frac{\sqrt{1 - \frac{4}{\pi^2}\varphi^2}}{\cos\varphi}\frac{1}{\sin\Theta}\sin\theta = \frac{\sqrt{1 - \frac{4}{\pi^2}\varphi^2}}{\cos\varphi}$$

which is not 1, so this projection is not equal-area. The Apian II projection is not widely used but several other projections are derived from it.

Let's now introduce a new variable, χ (see Figure ???), that is

$$\sin\chi = \frac{2\varphi}{\pi}$$

We can use it in the projection formulas:

$$x = \lambda \sqrt{1 - \frac{4}{\pi^2}\varphi^2} = \lambda\sqrt{1 - \sin^2\chi} = \lambda\cos\chi$$

$$y = \varphi = \frac{\pi}{2}\sin\chi$$

The Mollweide projection

Let's renumber the latitudes of Apian II to get an equal-area projection.

First, we have to apply a *c* reduction factor in the projection formulas to make the full grid area equal to the sphere surface:

$$x = c\,\lambda\cos\chi$$

$$y = c\frac{\pi}{2}\sin\chi$$

The projection outline is an ellipse with $c\frac{\pi}{2}$ and $c\pi$ half-axes. Therefore its area is:

$$A_{ellipse} = c\frac{\pi}{2}\,c\pi\,\pi = \frac{c^2\pi^2}{2}\pi$$

while the surface of the unit sphere is:

$$A_{sphere} = 4\pi$$

$A_{ellipse} = A_{sphere}$, so

$$\frac{c^2\pi^2}{2}\pi = 4\pi$$

which makes $c = \sqrt{\frac{8}{\pi^2}} = \sqrt{2}\frac{2}{\pi}$.

Now let's substitute χ with $\psi(\varphi)$:

$$x = c\,\lambda\cos\psi$$

$$y = c\frac{\pi}{2}\sin\psi$$

The surface of a spherical belt between the Equator and the latitude φ

$$A_{spherical} = 2\pi\sin\varphi$$

The area in the projection between the Equator and the parallel defined by ψ:

$$A_{projected} = 4(\sin\psi\cos\psi + \psi)$$

$A_{projected} = A_{spherical}$, so

$$2\pi\sin\varphi = 4(\sin\psi\cos\psi + \psi)$$

simplified:

$$\pi\sin\varphi = \sin 2\psi + 2\psi$$

Unfortunately this is again an implicit function, $\psi(\varphi)$ cannot be expressed, but the value of ψ can be estimated for any φ using a suitable numerical method.

To check that using this substitution we really get an equal-area projection we need to express the $\frac{d\psi}{d\varphi}$ derivative first by deriving the implicit function of $\psi(\varphi)$:

$$\frac{d}{d\varphi}\pi\,sin\,\varphi = \frac{d}{d\varphi}sin\,2\psi + 2\psi$$

$$\pi\cos\varphi = 2\cos 2\psi\frac{d\psi}{d\varphi} + 2\frac{d\psi}{d\varphi}$$

$$\frac{d\psi}{d\varphi} = \frac{\pi\cos\varphi}{2(\cos 2\psi + 1)} = \frac{\pi\cos\varphi}{2(2\cos^2\psi)} = \frac{\pi\cos\varphi}{4\cos^2\psi}$$

therefore

$$h = \frac{\frac{\partial x}{\partial\lambda}}{\cos\varphi} = c\ \frac{\cos\psi}{\cos\varphi}$$

$$k = \frac{dy}{d\varphi}\frac{1}{\sin\Theta} = \ c\frac{\pi}{2}\cos\psi\frac{d\psi}{d\varphi}\frac{1}{\sin\Theta} = c\frac{\pi}{2}\cos\psi\frac{\pi\cos\varphi}{4\cos^2\psi}\frac{1}{\sin\Theta} = c\frac{\pi^2\cos\varphi}{8\cos\psi}\frac{1}{\sin\Theta}$$

$$\tau = hk\sin\Theta = \ c\ \frac{\cos\psi}{\cos\varphi}\ c\frac{\pi^2\cos\varphi}{8\cos\psi}\frac{1}{\sin\Theta}\sin\Theta = c^2\frac{\pi^2}{8} = 1$$

so the Mollweide projection is really equal-area.

This projection is widely used for world maps in atlases.

Eckert III projection

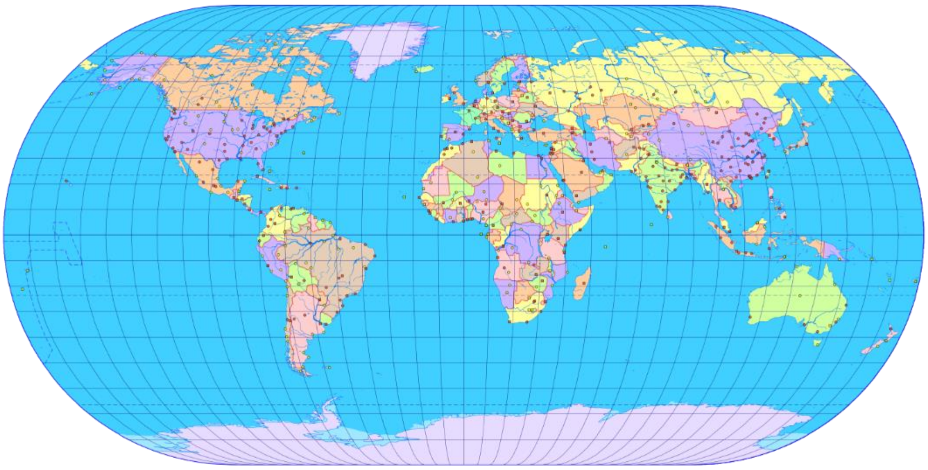

Figure 15. Eckert III projection

The Eckert III projection is a blended projection: the “average” of the Apian II and the Plate Carrée (Equirectangular) projections. We use the $\sin\chi = \frac{2\varphi}{\pi}$ substitution again, therefore $\varphi = \frac{\pi}{2}\sin\chi$:

Plate Carrée:

$$x_1 = \lambda$$
$$y_1 = \frac{\pi}{2}\sin\chi$$

Apian II:

$$x_1 = \lambda\cos\chi$$
$$y_1 = \frac{\pi}{2}\sin\chi$$

Blended:

$$x = \lambda\frac{1+\cos\chi}{2}$$
$$y = \frac{\pi}{2}\sin\chi$$

Similarly to Eckert V, the projection is reduced by a c constant in order to map the full sphere into a shape of the same area; therefore the Eckert II formulas are:

$$x = c\lambda\frac{1+\cos\chi}{2}$$

$$y = c\frac{\pi}{2}\sin\chi$$

The outline of the world map is two half-circles connected with straight lines having the same length as the half-circles diameter. The area of this shape is therefore the area of a square plus the area of a circle:

$$A_{proj} = c^2\pi^2 + \frac{c^2\pi^2}{4}\pi = c^2\pi^2\left(1+\frac{\pi}{4}\right)$$

which equals the sphere surface $A_{sphere} = 4\pi$, so

$$c = \frac{4}{\sqrt{\pi(4+\pi)}} \approx 0.844$$

This is a compromise (neither conformal nor equal-area) projection.

Eckert IV projection

The Eckert IV projection is gained by renumbering the latitudes of Eckert III to get an equal-area projection.

The process is similar to the one between Eckert V and VI: let's substitute χ by $\psi(\varphi)$ in the projection formulas of Eckert V:

$$x = c\lambda\frac{1+\cos\psi(\varphi)}{2}$$

$$y = c\frac{\pi}{2}\sin\psi(\varphi)$$

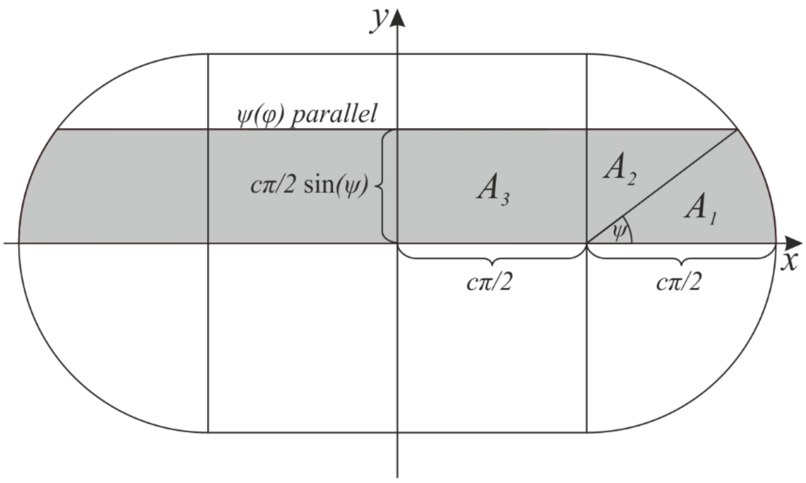

Figure 16. Shape of a spherical belt in Eckert IV projection

The $\psi(\varphi)$ function is defined by requiring the surface of the spherical belt between 0° and φ to be equal to the area of the shape lying between the Equator and the ψ parallel in the projection. This latter is observable on Figure …:

$$A_{projection} = 2(A_1 + A_2 + A_3) = 2\left(\frac{c^2\pi^2}{4}\frac{\psi}{2} + \frac{1}{2}c\frac{\pi}{2}\cos\psi\, c\frac{\pi}{2}\sin\psi + c\frac{\pi}{2}c\frac{\pi}{2}\sin\psi\right) =$$

$$= \frac{c^2\pi^2}{4}\left(\psi + \frac{\cos 2\psi}{2} + 2\sin\psi\right)$$

As the spherical surface of this belt is $A_{spherical} = 2\pi\sin\varphi$, and $A_{projection} = A_{spherical}$, we get

$$\frac{c^2\pi^2}{4}\left(\psi + \frac{\cos 2\psi}{2} + 2\sin\psi\right) = 2\pi\sin\varphi$$

simplified:

$$\frac{c^2\pi}{8}\left(\psi + \frac{\cos 2\psi}{2} + 2\sin\psi\right) = \sin\varphi$$

which is again an implicit function, defining the relation between ψ and φ.

Pseudocylindrical projections with straight meridians

Donis projection

The Donis projection is a pseudocylindrical projection with true scale Equator and central meridian. The pole is a point and all the meridians are straight lines (Figure 17). These requirements imply the following projection formulas:

$$y = \varphi$$

$$x = \lambda\left(1 - \frac{2}{\pi}|\varphi|\right)$$

The Donis projection was first used in the 14th-15th centuries.

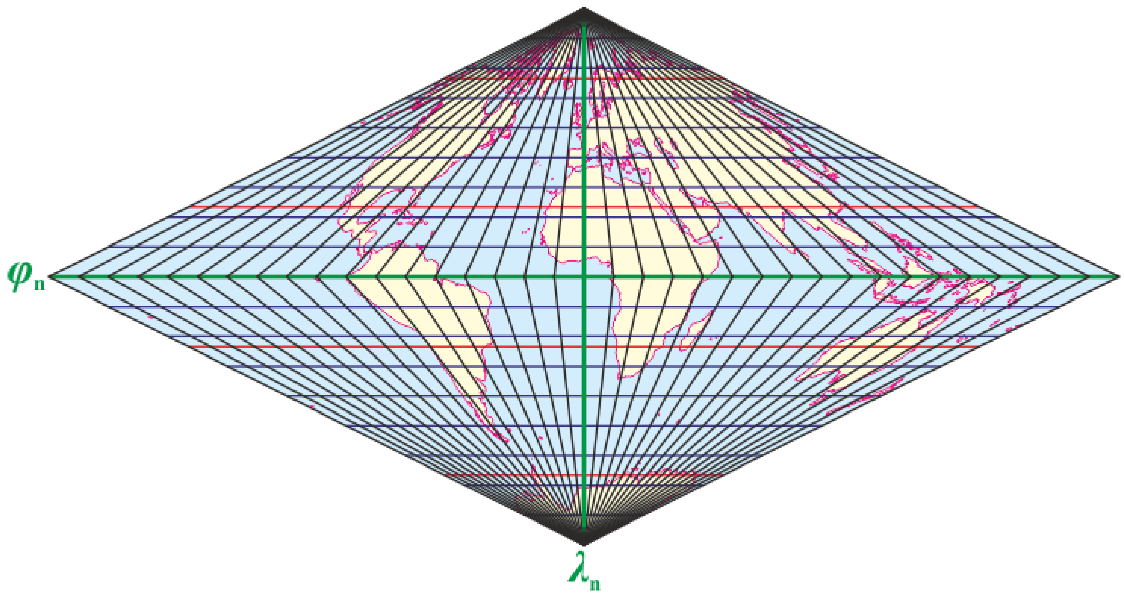

Figure 17. Donis projection

Collignon projection

The Collignon projection is produced by renumbering the latitudes of the Donis projection to make it equal-area.

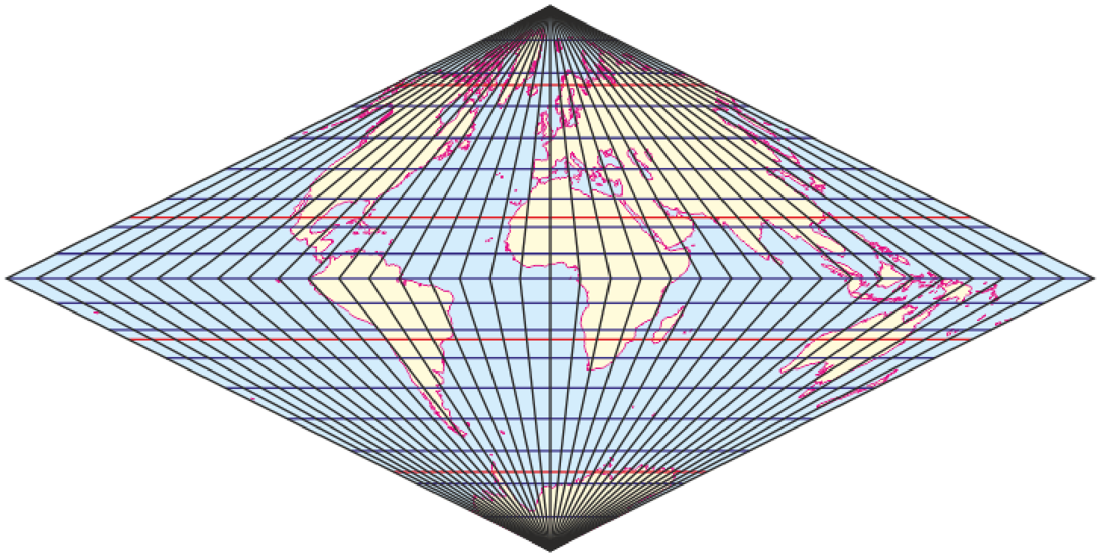

Figure 18. Collignon projection

Eckert I projection

The Eckert I is a blended projection: the formulas are gained by averaging the equations of the Plate Carrée and the Donis projection, and applying a reduction to make the shape of the full sphere have the same area as the sphere.

Eckert II projection

The Eckert II projection is created by renumbering the latitudes of Eckert I to make it equal area.

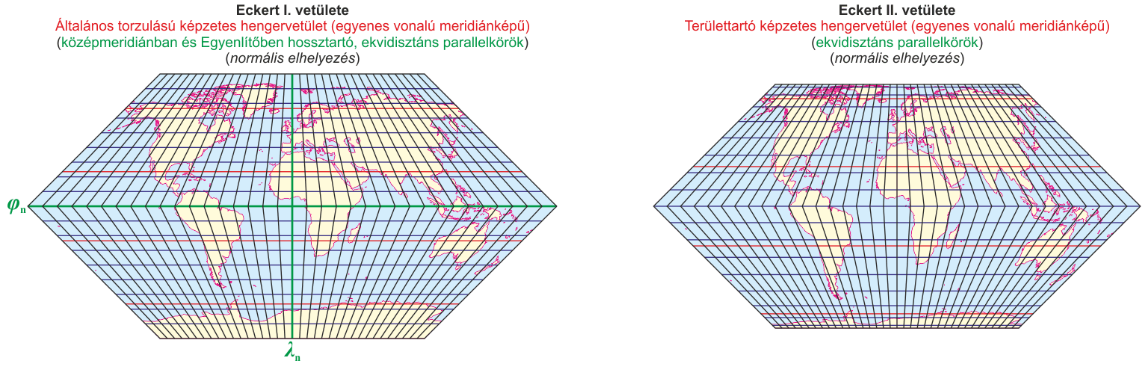

Figure 19. Eckert I and Eckert II projections

Pseudocylindrical composite projections

The distortions on different projections have different spatial distribution. E.g. some projections have lower distortions near the Equator while others are better around the poles.

Composite projections are constructed by using different projections for mapping different parts of the Earth. The advantage of this solution is that the composite projection can combine the best parts of the base projections, resulting a lower overall distortion.

Goode Homolosine projection

The most common way of creating a composite projection is to combine two pseudocylindrical projections: one for the equatorial areas and another for the higher latitudes, switching between them on a specified boundary latitude. As an example, we can combine the Sinusoidal and the Mollweide projections (Figure 20.).

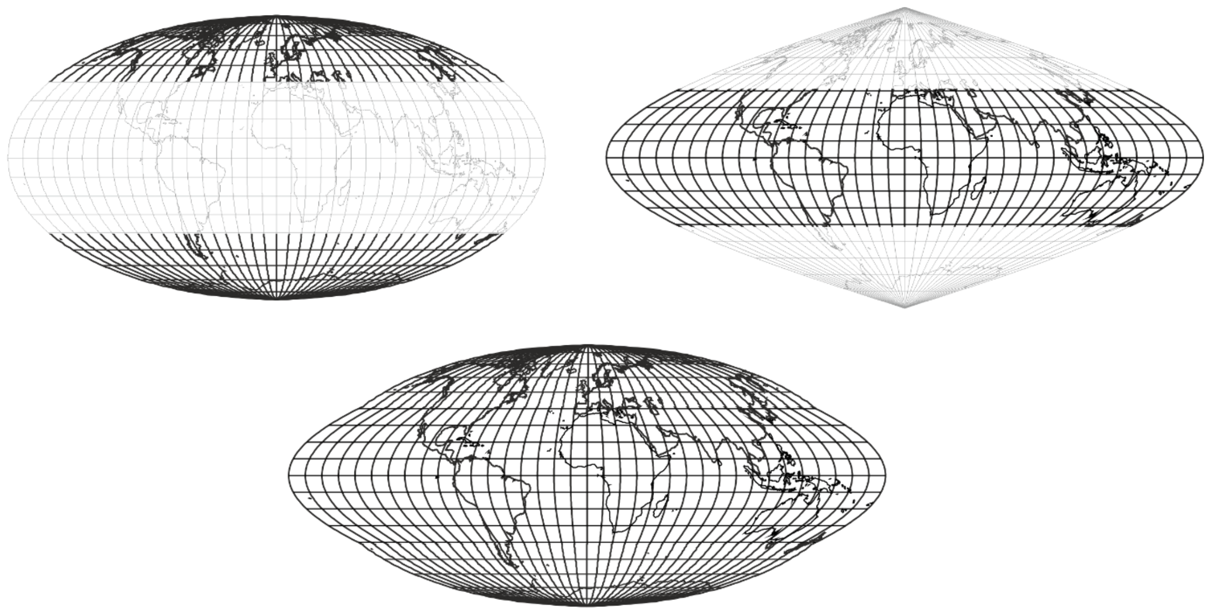

Figure 20. Creating a composite projection based on the Mollweide (upper left) and the Sinusoidal (upper right) projections

As all the parallels in the Sinusoidal projection are true scale, we have to find the φ_B true scale latitude of the Mollweide projection and join them at that latitude. This latitude can be easily calculated from the h scale factor along the parallels of the Mollweide projection using h = 1; which results in $\varphi_B \approx 40.73°$

This projection was introduced by John Paul Goode in 1923, with a modification: Goode interrupted the projection at several places, using multiple central meridians. The resulting projection can be observed on Figure 21. Goode called it "Homolosine" projection by mixing the name of the base projections (the Mollweide is also called "Homolographic" projection).

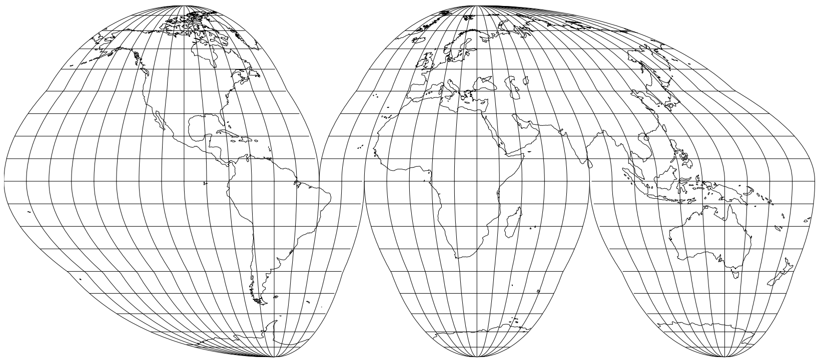

Figure 21. Goode's "Homolosine" projection

Érdi-Krausz projection

Goode's projection has a serious drawback: the meridians suffer a concave break where the two projection parts meet. The Hungarian cartographer, Érdi-Krausz György tried to solve this problem by using a Wagner transformed variant of the Sinusoidal projection (transformation parameters: p=0,4; q=0,6) for the equatorial regions. The two projection meets at the 60° or alternatively at the 70° latitude. A slight break of meridians is still observable, but in this case it is a convex break, which was usually smoothed by the cartographers. Another problem is that the size of the Mollweide parts has to be enlarged in order to fit the sinusoidal part. Therefore – although the projection is still called equal-area – the polar regions have different scale and it is always indicated on the maps using this projection.

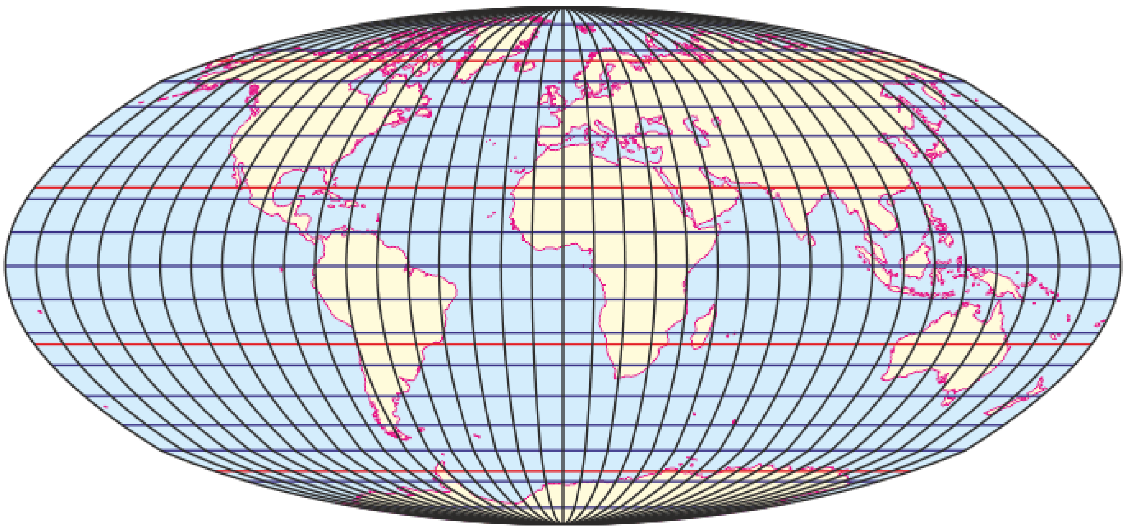

Figure 22. Érdi-Krausz projection