Pseudoconic projections

General properties

Pseudoconic projections in normal aspect share the following properties:

- Latitudes are circle arcs.
- The centres of these circles are along the central meridian (which is set to the y axis of the projection coordinate system)

As latitude lines are circle arcs, we can define a $p(\varphi)$ radius function. The position of the circle arcs can be defined either by the $c(\varphi)$ distance of the circle centre or the $t(\varphi)$ distance of the intersection of the circle and the y axis from the origin of the coordinate system (Figure …). Two of these values always define the third one as $c(\varphi) = t(\varphi) + p(\varphi)$.

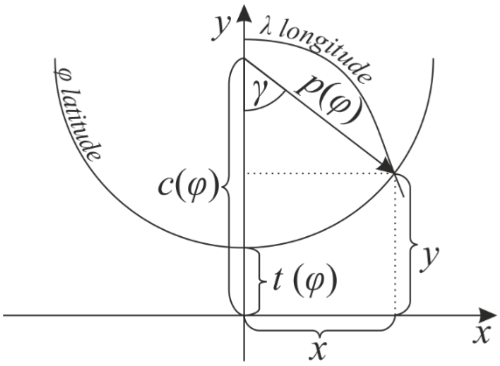

Figure … Spherical and projected coordinates of a point using pseudoconic projections.

The γ angle of a vector pointing from the centre of a latitude circle to an arbitrary (φ, λ) point is a function of φ and λ: $\gamma = \gamma(\varphi, \lambda)$.

Therefore a pseudoconic projection can be defined by three functions: its $p(\varphi)$ radius function, one from $c(\varphi)$ or $t(\varphi)$, and the $\gamma(\varphi, \lambda)$ function. The Cartesian coordinates then can be calculated as follows:

$$x = p(\varphi) \sin \gamma(\varphi, \lambda)$$

$$y = c(\varphi) - p(\varphi) \cos \gamma(\varphi, \lambda)$$

Distortions along the graticule lines

For developing the distortion formulas the partial derivatives of the above general projection formulas are needed:

$$\frac{\partial x}{\partial \lambda} = p \frac{\partial \gamma}{\partial \lambda} \cos \gamma$$

$$\frac{\partial y}{\partial \lambda} = p \frac{\partial \gamma}{\partial \lambda} \sin \gamma$$

$$\frac{\partial x}{\partial \varphi} = \frac{dp}{d\varphi} \sin \gamma + p \frac{\partial \gamma}{\partial \varphi} \cos \gamma$$

$$\frac{\partial y}{\partial \varphi} = \frac{dc}{d\varphi} - \frac{dp}{d\varphi} \cos \gamma + p \frac{\partial \gamma}{\partial \varphi} \sin \gamma$$

Therefore:

$$h = \frac{\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2}}{\cos\varphi} = \frac{p}{\cos\varphi}\frac{\partial \gamma}{\partial \lambda}$$

$$k = \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2} = \sqrt{\left(\frac{dp}{d\varphi}\sin\gamma + p\frac{\partial \gamma}{\partial \varphi}\cos\gamma\right)^2 + \left(\frac{dc}{d\varphi} - \frac{dp}{d\varphi}\cos\gamma + p\frac{\partial \gamma}{\partial \varphi}\sin\gamma\right)^2} = \cdots$$
$$= \sqrt{\left(\frac{dp}{d\varphi}\right)^2 + \left(p\frac{\partial \gamma}{\partial \varphi}\right)^2 + \left(\frac{dc}{d\varphi}\right)^2 - 2\frac{dc}{d\varphi}\frac{dp}{d\varphi}\cos\gamma + 2\frac{dc}{d\varphi}p\frac{\partial \gamma}{\partial \varphi}\sin\gamma}$$

$$\cot\Theta = \frac{\frac{\partial x}{\partial \lambda}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial \lambda}\frac{\partial y}{\partial \varphi}}{\frac{\partial x}{\partial \lambda}\frac{\partial y}{\partial \varphi} - \frac{\partial x}{\partial \varphi}\frac{\partial y}{\partial \lambda}}$$
$$= \frac{p\frac{\partial \gamma}{\partial \lambda}\cos\gamma\left(\frac{dp}{d\varphi}\sin\gamma + p\frac{\partial \gamma}{\partial \varphi}\cos\gamma\right) + p\frac{\partial \gamma}{\partial \lambda}\sin\gamma\left(\frac{dc}{d\varphi} - \frac{dp}{d\varphi}\cos\gamma + p\frac{\partial \gamma}{\partial \varphi}\sin\gamma\right)}{p\frac{\partial \gamma}{\partial \lambda}\cos\gamma\left(\frac{dc}{d\varphi} - \frac{dp}{d\varphi}\cos\gamma + p\frac{\partial \gamma}{\partial \varphi}\sin\gamma\right) - p\frac{\partial \gamma}{\partial \lambda}\sin\gamma\left(\frac{dp}{d\varphi}\sin\gamma + p\frac{\partial \gamma}{\partial \varphi}\cos\gamma\right)}$$
$$= \cdots = \frac{p\frac{\partial \gamma}{\partial \varphi} + \frac{dc}{d\varphi}\sin\gamma}{\frac{dc}{d\varphi}\cos\gamma - \frac{dp}{d\varphi}}$$

From this,

$$\sin\Theta = \frac{1}{\sqrt{1+\cot^2\Theta}} = \frac{1}{\sqrt{1 + \frac{\left(p\frac{\partial \gamma}{\partial \varphi} + \frac{dc}{d\varphi}\sin\gamma\right)^2}{\left(\frac{dc}{d\varphi}\cos\gamma - \frac{dp}{d\varphi}\right)^2}}} = \frac{\frac{dc}{d\varphi}\cos\gamma - \frac{dp}{d\varphi}}{\sqrt{\left(\frac{dc}{d\varphi}\cos\gamma - \frac{dp}{d\varphi}\right)^2 - \left(p\frac{\partial \gamma}{\partial \varphi} + \frac{dc}{d\varphi}\sin\gamma\right)^2}}$$
$$= \frac{\frac{dc}{d\varphi}\cos\gamma - \frac{dp}{d\varphi}}{k}$$

so

$$k = \frac{\frac{dc}{d\varphi}\cos\gamma - \frac{dp}{d\varphi}}{\sin\Theta}$$

“Real” pseudoconic projections

A more strict classification of projections use the term *pseudoconic* on projections that have **concentric** circle arcs as latitudes. It means, that $c(\varphi)$ is constant, therefore the formulas for k and $\cot\Theta$ are simpler:

$$k = \frac{-\frac{dp}{d\varphi}}{\sin\Theta}$$

$$\cot\Theta = -p\frac{\frac{\partial\gamma}{\partial\varphi}}{\frac{dp}{d\varphi}}$$

Bonne Projection

The most common example of this group is the Bonne Projection. This projection has true scale parallels ($h = 1$), and true scale central meridian ($k_{\lambda=0} = 1$). As $\sin\Theta = 1$, when $\lambda = 0$ (the central meridian is always an axis of symmetry, so the parallels cross it at right angle), $k_{\lambda=0} = 1$ means that

$$-\frac{dp}{d\varphi} = 1$$

so

$$p = const. - \varphi$$

True scale parallels means

$$h = \frac{p}{\cos\varphi}\frac{\partial\gamma}{\partial\lambda} = 1$$

Expressing $\frac{\partial\gamma}{\partial\lambda}$:

$$\frac{\partial\gamma}{\partial\lambda} = \frac{\cos\varphi}{p}$$

which implies

$$\gamma = \lambda\frac{\cos\varphi}{p}$$

We can set constant part of $p(\varphi)$ to have a given φ_s standard parallel free of distortions. As all parallels are true scale, we only have to make sure that Θ is 90° (i. e. $\cot\Theta = 0$) all along φ_s:

$\cot\Theta = -p\frac{\frac{\partial\gamma}{\partial\varphi}}{\frac{dp}{d\varphi}} = 0$, when $\varphi = \varphi_s$.

As $\frac{\partial\gamma}{\partial\varphi} = \lambda\frac{-p\sin\varphi - \frac{dp}{d\varphi}\cos\varphi}{p^2}$ and $\frac{dp}{d\varphi} = -1$,

$$\cot\Theta = -p\frac{\frac{\partial\gamma}{\partial\varphi}}{\frac{dp}{d\varphi}} = p\lambda\frac{-p\sin\varphi + \cos\varphi}{p^2} = \lambda\frac{-p\sin\varphi + \cos\varphi}{p}$$

this can be zero when the nominator is zero: $-p\sin\varphi + \cos\varphi = 0$. Substituting $\varphi = \varphi_s$, we get

$-p\sin\varphi + \cos\varphi = (\varphi_s - const.)\sin\varphi_s + \cos\varphi_s = 0$ which lead to

$$const. = \frac{\cos\varphi_s}{\sin\varphi_s} + \varphi_s = \cot\varphi_s + \varphi_s$$

So finally:

$$p = \cot\varphi_s + \varphi_s - \varphi$$

$$\gamma = \lambda \frac{\cos\varphi}{p} = \lambda \frac{\cos\varphi}{\cot\varphi_s + \varphi_s - \varphi}$$

If we examine the area distortions, as $h = 1$ and $k = \frac{-\frac{dp}{d\varphi}}{\sin\Theta} = \frac{1}{\sin\Theta}$ (as $\frac{dp}{d\varphi} = -1$), we get

$$\tau = hk\sin\Theta = 1\frac{1}{\sin\Theta}\sin\Theta = 1$$

which means that this projection is **equal area**.

The Bonne projection was developed in the 16th century. It is used for small-scale regional maps such as continent parts (e.g. West Africa). Changing φ_s changes the shape of the map. A special case, $\varphi_s = 90°$ results in a heart-shaped world map (Cardioid or Werner-Stabius projection).

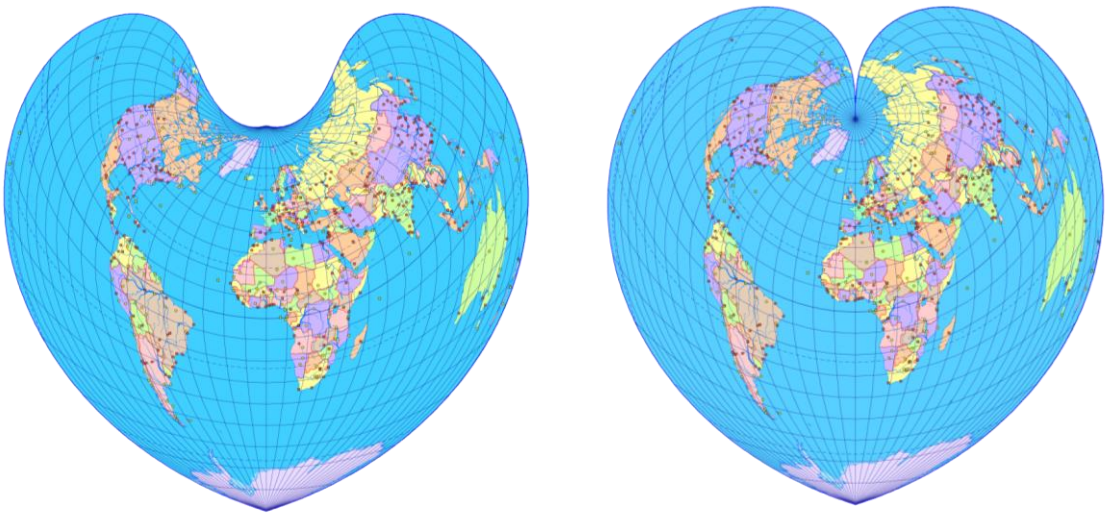

Figure .... Bonne projection with $\varphi_s = 45°$ (left) and $\varphi_s = 90°$ (right).

… to be continued!!!!