Distortions along graticule lines

As the projection formulas required to be twice continuously differentiable, their values can be locally estimated by linear estimation using their partial derivatives:

$$\Delta x = \frac{\partial x}{\partial \lambda}\Delta\lambda + \frac{\partial x}{\partial \varphi}\Delta\varphi$$

$$\Delta y = \frac{\partial y}{\partial \lambda}\Delta\lambda + \frac{\partial y}{\partial \varphi}\Delta\varphi$$

where Δx and Δy is the difference in projected coordinates caused by a little $\Delta\varphi$ change in latitude and $\Delta\lambda$ in longitude.

Scale factors along the meridians and the parallels

Let's move on a unit sphere from any point P along the local meridian to P_m, and along the local parallel to P_p, denoting these distances as Δt_m and Δt_p, respectively, and observe the projected image (Figure 6).

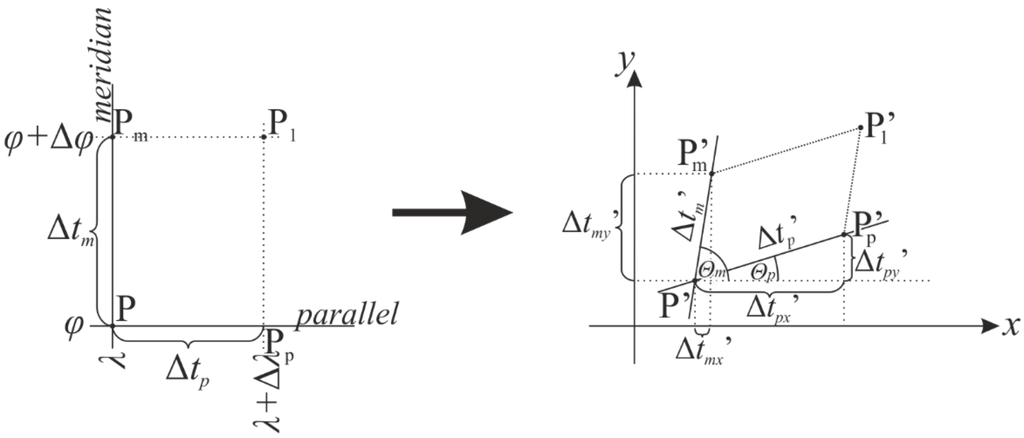

Figure 6. Little distances along the meridian and the parallel on the globe (left) and their projected image (right)

As we are on a unit sphere, $\Delta t_m = \Delta\varphi$ and $\Delta t_p = \Delta\lambda \cos\varphi$. The projected images of these distances ($\Delta t_m'$ and $\Delta t_p'$) can be calculated from their horizontal and vertical components ($\Delta t_{mx}'$, $\Delta t_{my}'$, $\Delta t_{px}'$ and $\Delta t_{py}'$) that are estimeted as the product of $\Delta\varphi$ or $\Delta\lambda$ and the correspondong partial derivative:

$$\Delta t'_{mx} = \frac{\partial x}{\partial \varphi}\Delta\varphi$$

$$\Delta t'_{my} = \frac{\partial y}{\partial \varphi}\Delta\varphi$$

$$\Delta t'_{px} = \frac{\partial x}{\partial \lambda}\Delta\lambda$$

$$\Delta t'_{py} = \frac{\partial y}{\partial \lambda}\Delta\lambda$$

Therefore, using the Pythagorean theorem,

$$\Delta t'_m = \sqrt{\Delta t'^2_{mx} + \Delta t'^2_{my}} = \sqrt{\left(\frac{\partial x}{\partial \varphi}\Delta\varphi\right)^2 + \left(\frac{\partial y}{\partial \varphi}\Delta\varphi\right)^2} = \Delta\varphi\sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2}$$

$$\Delta t'_p = \sqrt{\Delta t'^2_{px} + \Delta t'^2_{py}} = \sqrt{\left(\frac{\partial x}{\partial \lambda}\Delta\lambda\right)^2 + \left(\frac{\partial y}{\partial \lambda}\Delta\lambda\right)^2} = \Delta\lambda\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2}$$

Now we can calculate the *k* scale factor along the meridian as

$$k = \lim_{\Delta\varphi\to 0}\frac{\Delta t'_m}{\Delta t_m} = \frac{\Delta\varphi\sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2}}{\Delta\varphi} = \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2}$$

and similarly the *h* scale factor along the parallel as

$$h = \lim_{\Delta\varphi\to 0}\frac{\Delta t'_p}{\Delta t_p} = \frac{\Delta\lambda\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2}}{\Delta\lambda\cos\varphi} = \frac{\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2}}{\cos\varphi}$$

The angle of the projected graticule lines

The Θ angle of the projected graticule lines also can be calculated as the difference of Θ_m and Θ_p. The sine and cosine of these angles can be read from the figure:

$$\cos\Theta_p = \frac{\frac{\partial x}{\partial \lambda}\Delta\lambda}{\Delta t'_p}$$

$$\sin\Theta_p = \frac{\frac{\partial y}{\partial \lambda}\Delta\lambda}{\Delta t'_p}$$

$$\cos\Theta_m = \frac{\frac{\partial x}{\partial \varphi}\Delta\varphi}{\Delta t'_m}$$

$$\sin\Theta_m = \frac{\frac{\partial y}{\partial \varphi}\Delta\varphi}{\Delta t'_m}$$

Therefore cot Θ can be expressed as:

$$\cot\Theta = \frac{\cos\Theta}{\sin\Theta} = \frac{\cos(\Theta_m - \Theta_p)}{\sin(\Theta_m - \Theta_p)} = \frac{\sin\Theta_m \sin\Theta_p + \cos\Theta_m\cos\Theta_p}{\sin\Theta_m\cos\Theta_p - \cos\Theta_m\sin\Theta_p} = \frac{\frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial \lambda} + \frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \lambda}}{\frac{\partial y}{\partial \varphi}\frac{\partial x}{\partial \lambda} - \frac{\partial x}{\partial \varphi}\frac{\partial y}{\partial \lambda}}$$