Connection between the Tissot indicatrix and distortions along the graticule lines

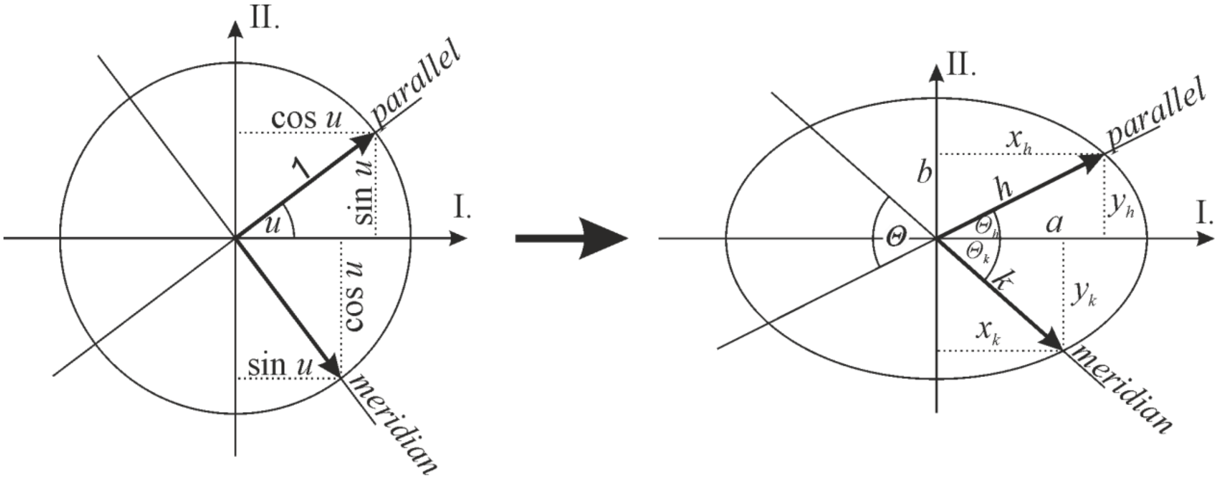

Figure 7. A small unit circle on the globe (left) and its projected image, the Tissot indicatrix (right). The coordinate axes are the principal directions.

A small unit circle on the globe is projected into an ellipse with half axes a and b (the measures of the Tissot indicatrix). If we also draw unit vectors along the local parallel and meridian lines, their projected lengths are h and k, the scale factors along the graticule lines. Observing Figure 7, we can notice that the right hand figure is a affine transformed version of the left hand one, therefore x_h, y_h, x_k and y_k can be calculated by multiplying sin u and cos u by a and b:

$$x_h = a\cos u$$

$$y_h = b\sin u$$

$$x_k = a\sin u$$

$$y_k = b\cos u$$

Now we can express h and k as:

$$h^2 = x_h^2 + y_h^2 = a^2\cos^2 u + \ b^2\sin^2 u$$

$$k^2 = x_k^2 + y_k^2 = a^2\sin^2 u + \ b^2\cos^2 u$$

therefore

$$h^2 + k^2 = a^2(\cos^2 u + \sin^2 u) + \ b^2(\cos^2 u + \sin^2 u) = a^2 + b^2$$

additionally,

$$\sin\Theta = \sin(\Theta_k + \Theta_h) = \sin\Theta_k\cos\Theta_h + \cos\Theta_k\sin\Theta_h = \frac{y_k}{k}\frac{x_h}{h} + \frac{x_k}{k}\frac{y_h}{h} = \frac{y_k x_h + x_k y_h}{hk}$$

so

$$hk\sin\Theta = y_k x_h + x_k y_h = b\cos u\, a\cos u + a\sin u\, b\sin u = ab(\cos^2 u + \sin^2 u) = ab$$

To express a and b with h and k, first let's write $(a+b)^2$ and $(a-b)^2$:

$$(a+b)^2 = a^2 + b^2 + 2ab = h^2 + k^2 + 2hk\sin\Theta$$

$$(a-b)^2 = a^2 + b^2 - 2ab = h^2 + k^2 + 2hk \sin\Theta$$

therefore

$$a + b = \sqrt{h^2 + k^2 + 2hk \sin\Theta}$$

$$a - b = \sqrt{h^2 + k^2 - 2hk \sin\Theta}$$

We can express *a* by adding the two equations above and *b* by subtracting them:

$$a = \frac{\sqrt{h^2 + k^2 + 2hk \sin\Theta} + \sqrt{h^2 + k^2 - 2hk \sin\Theta}}{2}$$

$$b = \frac{\sqrt{h^2 + k^2 + 2hk \sin\Theta} - \sqrt{h^2 + k^2 - 2hk \sin\Theta}}{2}$$