Azimuthal (planar) projections

General properties

Azimuthal projections in normal aspect share the following properties:

- Parallels are concentric full circles
- Meridians are straight lines, crossing in one point, the pole
- Parallels and meridians are perpendicular to each other
- Angles at the pole are mapped without distortion

For azimuthal projections in oblique or transverse aspect, a meta coordinate system can be defined with its metapole matching the projection centre, and the properties above are true to the metaparallels, metameridians and the metapole.

Due to these properties, azimuthal projections can be defined by their radius function: a $p(\beta)$ function that gives the p radius of a parallel circle with β polar distance ($\beta = 90° - \varphi$), see Figure 8.

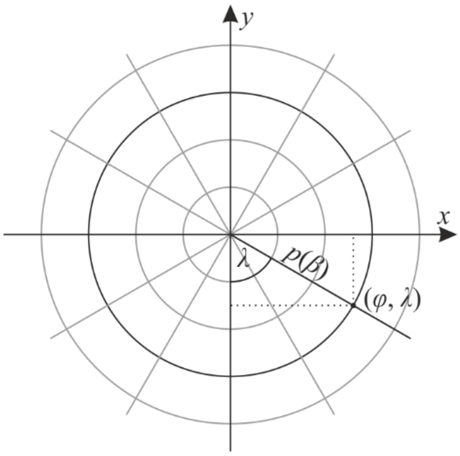

Figure 8. Azimuthal projections in normal (polar) aspect.

The general projection equations of a normal aspect azimuthal projection are therefore:

$$x = p(\beta)\sin\lambda$$

$$y = -p(\beta)\cos\lambda$$

For any other aspect, the (β^*, λ^*) meta-coordinates can be calculated from (φ, λ) and substituted into the equations above instead of the real β and λ.

Distortions along the graticule lines

The general formulas for the scale factors along the parallels (h) and the meridians (k) can be simplified after expressing the partial derivatives from the projection formulas above, knowing, that $\frac{d\beta}{d\varphi} = -1$, because $\beta = 90° - \varphi$.

$$\frac{\partial x}{\partial \varphi} = \frac{dp}{d\varphi}\sin\lambda = \frac{dp}{d\beta}\frac{d\beta}{d\varphi}\sin\lambda = -\frac{dp}{d\beta}\sin\lambda$$

$$\frac{\partial y}{\partial \varphi} = -\frac{dp}{d\varphi}\cos\lambda = \frac{dp}{d\beta}\cos\lambda$$

$$\frac{\partial x}{\partial \lambda} = p(\beta)\cos\lambda$$

$$\frac{\partial y}{\partial \lambda} = p(\beta)\sin\lambda$$

therefore:

$$h = \frac{\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2}}{\cos\varphi} = \frac{p(\beta)\sqrt{\cos^2\lambda + \sin^2\lambda}}{\cos\varphi} = \frac{p(\beta)}{\cos\varphi} = \frac{p(\beta)}{\sin\beta}$$

$$k = \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2} = \frac{dp}{d\beta}\sqrt{\sin^2\lambda + \cos^2\lambda} = \frac{dp}{d\beta}$$

The intersection angle of the graticule lines, $\Theta = 90°$ always, as they are perpendicular by definition. As $a^2 + b^2 = h^2 + k^2$, and $ab = hk\sin\Theta = hk$, either $a = h,\ b = k$ or $a = k,\ b = h$ (a is always the larger one). Therefore, the criteria of conformality (no angular distortions), $a = b$ implies $h = k$.

The area distortion factor $\tau = hk\sin\Theta = hk$, so the criteria of an equal-area projection is $h = \frac{1}{k}$.

Perspective azimuthal projection (Vertical Perspective)

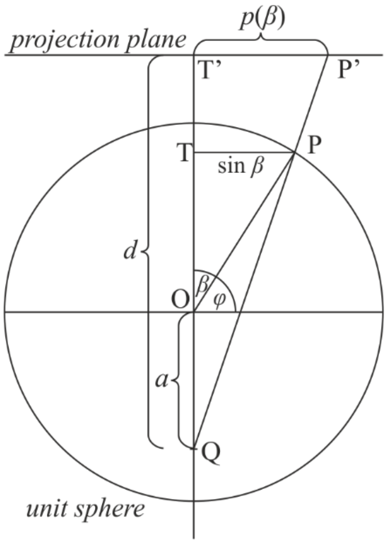

Figure 9?. Perspective azimuthal projection

Let's map the *unit sphere* to a plane by using central perspective projection. The configuration can be seen on Figure 9: the point of perspective (*Q*) is on the axis of rotation, its distance from the sphere centre (*O*) is a. The projection plane is perpendicular to the axis and its distance from *Q* is *d*. *P* is an arbitrary point on the sphere surface with a polar distance *β*. *P'*, the figure of *P* on the projection plane is constructed as the intersection of the plane and the line connecting *Q* and *P*. *T'* is the intersection of the axis and the plane, while *T* is the point on the axis from where *P* is at right angle. As the sphere is a unit sphere (its radius is 1), $TP = \sin\beta$, and $OT = \cos\beta$. The *T'P'* distance becomes the p radius of the parallel of polar distance *β*, i. e. the *p(β)* radius function.

As the QTP and the QT'P' triangles are similar, $\frac{T'P'}{QT'} = \frac{TP}{QT}$. After substitution, we get:

$$\frac{p}{d} = \frac{\sin\beta}{a + \cos\beta}$$

$$p = \frac{d\sin\beta}{a + \cos\beta}$$

which is the radius function of the perspective azimuthal projection.

In GIS systems it is called Vertical Perspective (to differentiate it from Tilted perspective, where the projection plane is tilted). There are two variants: Near-side and Far-side Vertical Perspective projections. The only difference is that in the Near-side variant Q is at the opposite side of the plane.

The scale factors along the meridians and parallels are the following:

$$h = \frac{p(\beta)}{\sin\beta} = \frac{d}{a + \cos\beta}$$

$$k = \frac{dp}{d\beta} = \frac{d\cos\beta\ \ (a + \cos\beta) + d\sin^2\beta}{(a + \cos\beta)^2} = \frac{d + ad\cos\beta}{(a + \cos\beta)^2}$$

The Vertical perspective projection in its general form is rarely used (although it is worth mentioning that any vertical view of geobrowsers like Google Earth is in this projection), but there are three special cases, that have unique properties.

Gnomonic Projection: $a = 0,\ d = 1$

In this projection the point of perspective is the sphere centre. (The name “gnomonic” refers to the “gnomes [dwarves] point of view”.) The radius function is:

$$p = \frac{d\sin\beta}{a + \cos\beta} = \frac{\sin\beta}{\cos\beta} = \tan\beta$$

the scale factors along the graticule: $h = \frac{p(\beta)}{\sin\beta} = \frac{1}{\cos\beta}$ and $k = \frac{dp}{d\beta} = \frac{1}{\cos^2\beta}$.

$h \neq k$ and $h \neq \frac{1}{k}$, so this projection is neither conformal nor equal area.

Only regions smaller than a hemisphere can be mapped with this projection, as p grows to infinity as β approaches 90°. The distortions also grow fast with the distance from the projection centre. The projection, however, has an advantage as well: all the great circles (orthodromes) are mapped as straight lines. This is obvious as the centre of the great circles is identical to the sphere centre, therefore to the point of perspective, and any circle seems as a line from its centre.

Orthographic projection: $a \to \infty, d = a + 1$

In this projection the point of perspective is in infinite distance, therefore the projection rays are parallel to the axis of rotation. The radius function becomes $p = \sin\beta$. The scales along the graticule:

$$h = \frac{p(\beta)}{\sin\beta} = \frac{\sin\beta}{\sin\beta} = 1$$

$$k = \frac{dp}{d\beta} = \cos\beta$$

This projection is again neither conformal nor equal-area, but has true scale (meta-)parallels. The maximum area mappable is a hemisphere. This projection is often used (in transverse aspect) for mapping other planets of the Solar System, as it is very similar to the picture one can see in a telescope.

Stereographic projection: $a = 1,\ d = 2$

In this projection, the point of perspective is the opposite pole (or, in oblique aspect, the opposite point of the sphere). The radius function is:

$$p = \frac{2\sin\beta}{1+\cos\beta} = \frac{4\sin\frac{\beta}{2}\cos\frac{\beta}{2}}{1+\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}} = \frac{4\sin\frac{\beta}{2}\cos\frac{\beta}{2}}{2\cos^2\frac{\beta}{2}} = 2\tan\frac{\beta}{2}$$

As $\beta \to \infty$ if $\beta \to 180°$, the whole sphere cannot be mapped. It is usually applied on areas not larger than a hemisphere.

The scales along the graticule:

$$h = \frac{p(\beta)}{\sin\beta} = \frac{1}{\cos^2\beta}$$

$$k = \frac{dp}{d\beta} = \frac{1}{\cos^2\beta}$$

$h = k$, so this projection is *conformal*. This property makes this projection very popular, especially for mapping polar regions. It was also used in topographic mapping, e. g. in Hungary.

Azimuthal projection with true scale meridians (Azimuthal Equidistant Projection)

Let's construct an azimuthal projection with true scale meridians. This criteria means that $k = \frac{dp}{d\beta} = 1$, which results in the radius function $p = \beta\ (+constant)$. The constant has to be zero in order to have the pole as a point, not a circle, therefore $p = \beta$.

The scale factor along the parallels:

$$h = \frac{p(\beta)}{\sin\beta} = \frac{1}{\sin\beta}$$

This projection is named "Postel projection" in some countries after Guillaume Postel, who used it for a map in 1581. It is mostly used in polar aspect for mapping polar regions (or the whole world, as it is in the flag of the United Nations). In traditional globe making, the polar caps are also created in this projection.

It is also useful when distances from a given point have to be mapped true scale. In this case it has to be used in oblique form, with the projection centre set to that point.

Azimuthal Equal-Area projection

To construct an azimuthal equal-area projection, let's start from the criteria $h = \frac{1}{k}$. As $h = \frac{p(\beta)}{\sin\beta}$, and $k = \frac{dp}{d\beta}$, this implies the following differential equation:

$$\frac{dp}{d\beta} = \frac{\sin\beta}{p}$$

Its solution:

$$\int p\,dp = \int \sin\beta\,d\beta$$

$$\frac{p^2}{2} = -\cos\beta + c$$

where c is an arbitrary constant. As we would like to map the pole as a point, so let $\beta = 0$ when $\beta = 0$. It implies $c = 1$, so $p = \sqrt{2 - 2\cos\beta}$.

Using half angles, this formula can be simplified:

$$p = \sqrt{2 - 2\cos\beta} = \sqrt{2 - 2\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right)} = \sqrt{4\sin^2\frac{\beta}{2}} = 2\sin\frac{\beta}{2}$$

The full name of the projection is Lambert Azimuthal Equal-Area projection after the Swiss-Elsatian mathematician *Johann Heinrich Lambert*, who announced it in 1772.

Being an equal-area projection, it is often used for small-scale mapping regions or continents with more or less similar north–south and east–west dimensions.