# Cylindrical projections

## General properties

Cylindrical projections in normal aspect share the following properties:

- Parallels (latitudes) are straight parallel lines
- Meridians are straight parallel lines
- Parallels and meridians are perpendicular to each other
- The meridians are equally spaced
- The graticule is symmetric to the Equator

For cylindrical projections in oblique or transverse aspect, a meta coordinate system can be defined with its metapole in 90° distance of the projection centre (The projection centre is in the intersection of the meta-Equator and the meta prime meridian). In this case the properties above are true to the metaparallels and metameridians.

These properties imply the following general projection equations for cylindrical projections in normal aspect:

$$x = c(\lambda - \lambda_0)$$

$$y = y(\varphi)$$

$\lambda_0$ is the central meridian. $y$ is an odd function: $y(-\varphi) = -y(\varphi)$. The $c$ constant can be chosen arbitrarily, but generally $0 < c \leq 1$.

### Distortions along the graticule lines

Note that $x$ only depends on the longitude, while $y$ only depends on the latitude, therefore $\frac{\partial x}{\partial \varphi} = 0$ and $\frac{\partial y}{\partial \lambda} = 0$. This simplifies the formulas for $h$ and $k$:

$$h = \frac{\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2}}{\cos\varphi} = \frac{\sqrt{\left(\frac{dx}{d\lambda}\right)^2}}{\cos\varphi} = \frac{\frac{dx}{d\lambda}}{\cos\varphi} = \frac{c}{\cos\varphi}$$

$$k = \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2} = \sqrt{\left(\frac{dy}{d\varphi}\right)^2} = \frac{dy}{d\varphi}$$

Let's use the substitution $c = \cos\varphi_n$. Then $h = \frac{\cos\varphi_n}{\cos\varphi}$, which means that the scale along the parallel $\varphi_n$ is one, so $\varphi_n$ is the *true scale* (or *standard*) *parallel*.

The projection equations for the normal aspect therefore become:

$$x = \cos\varphi_n\,(\lambda - \lambda_0)$$

$$y = y(\varphi)$$

The various types of cylindrical projections only differ in their $y(\varphi)$ projection formula.

For any other aspect, the $(\varphi^*, \lambda^*)$ meta-coordinates can be calculated from $(\varphi, \lambda)$ and substituted into the equations above instead of the real $\varphi$ and $\lambda$.

The intersection angle of the graticule lines, $\Theta = 90°$ always, as they are perpendicular by definition. So, just like in the case of azimuthal projections, either $a = h,\ b = k$ or $a = k,\ b = h$. The criteria of comformality is $h = k$, while the criteria of an equal-area projection is $h = \frac{1}{k}$.

## Central perspective cylindrical projection

*Note: This projection has no practical use due to its high distortions, but briefly discussed here because this is the "original" cylindrical projection; the general properties, and the name "cylindrical" comes from this projection.*

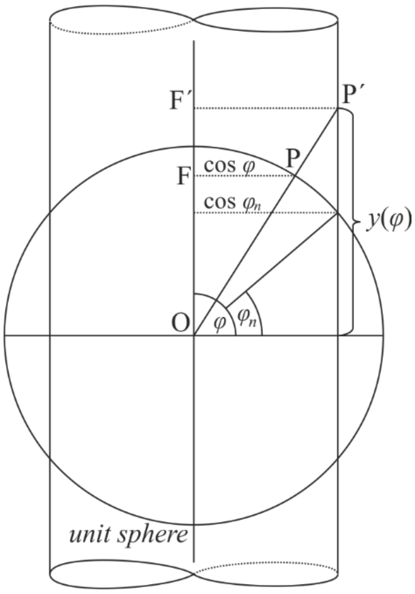


*Figure 10. Construction of central perspective cylindrical projection.*

Let's place a cylinder around a unit sphere with its axis matching the axis of the globe (Figure 10). The radius of the cylinder is not larger than the radius of the globe. Now let's project the surface of the sphere to the cylinder from the sphere centre. The distance of the projected point from the plane of the Equator becomes the $y(\varphi)$ coordinate. Due to the similarity of the triangles OFP and OF'P', the following equation can be constructed:

$$\frac{OF'}{F'P'} = \frac{OF}{FP}$$

where $OF' = y(\varphi)$, $F'P' = \cos\varphi_n$, $OF = \sin\varphi$ and $FP = \cos\varphi$, so

$$\frac{y(\varphi)}{\cos\varphi_n} = \frac{\sin\varphi}{\cos\varphi}$$

threrefore

$$y = \cos\varphi_n \tan\varphi$$

The perspective cylindrical projection can be constructed using a point of perspective differen from the sphere centre (but along the axis of rotation). These projections, however, are no longer symmetric to the Equator.

The projection is not often used, but it appears in the Russian "Атлас мира" (Atlas of the World).

## Cylindrical projection with true scale meridians

Let's start with the formula for true scale meridians:

$$k = \frac{dy}{d\varphi} = 1$$

This implies the formula: $y = \varphi + constant$, where the constant is zero, because $y(\varphi)$ has to be an odd function. Therefore the equations of this projection in normal aspect are:

$$x = \cos\varphi_n\,(\lambda - \lambda_0)$$

$$y = \varphi$$

The name of the projection is Equirectangular Projection. If the standard parallel is the Equator ($\varphi_n = 0$), its name is Plate Carrée or Geographic Projection. Notice that while GIS systems use metric coordinates in Equirectangular Projection, degrees are used in the case of Geographic Projection.

This projection is neither equal-area nor conformal. In normal aspect, it can be used for small-scale mapping equatorial countries, regions. It also can be used in oblique aspect for regions that are along a great circle. In this case the meta-Equator of the oblique aspect has to match that great circle.

The transverse form of the Geographic Projection is called Cassini or Cassini-Soldner Projection, and its ellipsoidal version was used in topographic mapping in the past. In traditional globe making, the printed globe segments (gores) are also in Cassini projection.

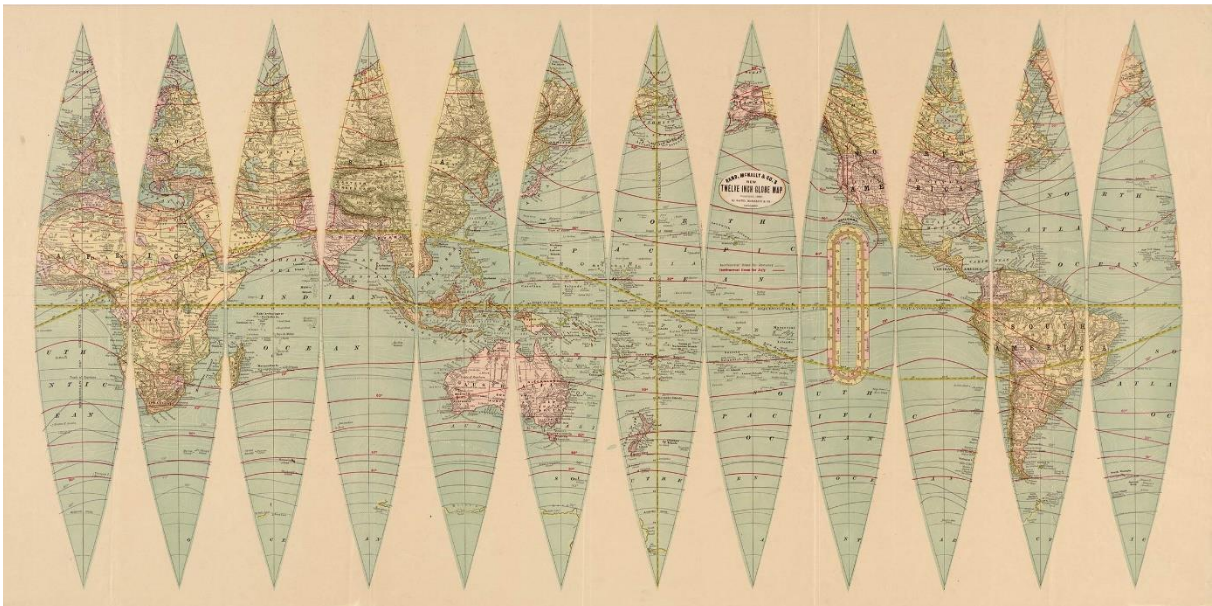

*Figure 11. Globe gore set in Cassini projection. (Rand McNally & Co., 1887)*

## Equal-area cylindrical projection

Now let's start from the criteria for an equal-area cylindrical projection: $h = \frac{1}{k}$.

As $h = \frac{\cos\varphi_n}{\cos\varphi}$ and $k = \frac{dy}{d\varphi}$, we get

$$\frac{dy}{d\varphi} = \frac{\cos\varphi}{\cos\varphi_n}$$

which means, that

$$y = \frac{\sin\varphi}{\cos\varphi_n}$$

The projection was developed by Lambert in 1772. Special versions were later introduced by others (Behrmann, $\varphi_n = 30°$, 1910; Gall–Peters, $\varphi_n = 45°$, 1855/1967). An interesting version is the "square world" created by Waldo Tobler, with the setting $\cos\varphi_n = \frac{1}{\sqrt{\pi}}$ (therefore $\varphi_n \approx 55°39'14''$). With this setting, the full globe is mapped to a square.

In GIS systems this projection is usually known as *Lambert Cylindrical Equal-Area*.

## Cylindrical conformal projection

To create a conformal projection, let's start with the criteria of conformality: $h = k$. As $h = \frac{\cos\varphi_n}{\cos\varphi}$ and $k = \frac{dy}{d\varphi}$, we get

$$\frac{dy}{d\varphi} = \frac{\cos\varphi_n}{\cos\varphi}$$

which leads to the formula:

$$y = \cos\varphi_n \ln\left[\tan\left(45° + \frac{\varphi}{2}\right)\right]$$

or

$$y = \cos\varphi_n \ln\left[\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\right]$$

depending on whether angles are counted in radians or degrees.

*Note: To solve this, we need the indefinite integral (a.k.a. primitive function) of* $\frac{1}{x}$*, which is* $\ln\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right] + const.$ *To prove it, let's derive it back:*

$$\frac{d}{dx}\ln\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right] = \frac{1}{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)}\frac{d}{dx}\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \frac{\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)}{\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)}\frac{1}{2\cos^2\left(\frac{\pi}{4} + \frac{x}{2}\right)}$$
$$= \frac{1}{2\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)} = \frac{1}{\sin\left(\frac{\pi}{2} + x\right)} = \frac{1}{\cos x}$$

This projection was introduced by Gerardus Mercator in 1569, and named after him as the Mercator Projection. It had an extremely important role in sea navigation, as this is the only projection that shows the *rhumb lines* or *loxodromes* (spherical curves with constant azimuth) as straight lines. Therefore, planning a sea route between two points was easy: a line connecting the points on Mercator's map was a loxodrome, so sailors only needed to keep the bearing of this line by a compass to keep the course.

As the loxodrome is usually not the shortest route between two points, with the emerge of global positioning techniques this projection lose its importance until the era of web maps. Nowadays most of web map services (like Google Maps) use a slightly modified version of this projection, called Web Mercator. In this projection, the WGS 84 ellipsoidal coordinates are directly used in the spherical formulas of the Mercator projection.

The ellipsoidal version of Mercator projection is extensively used in topographic mapping in transverse or oblique aspect. The transverse aspect is the basis of the Universal Transverse Mercator (UTM) and the Gauss-Krüger projection systems. The oblique version is the basis of the Swiss and also the Hungarian National Grid (EOV).