Conic Projections

General properties

Conic projections in normal aspect share the following properties:

- Parallels are concentric circle arcs
- Meridians are straight lines, crossing in one point, the pole
- Parallels and meridians are perpendicular to each other
- Angles at the pole are mapped with reduction: $\lambda' = n\lambda$ where $0 < n < 1$.

For conic projections in oblique or transverse aspect, a meta coordinate system can be defined with its metapole matching the projection centre, and the properties above are true to the metaparallels, metameridians and the metapole.

Due to these properties, conic projections can be defined by their $p(\beta)$ radius function (similarly to azimuthal projections) and the *n* angle proportion constant (Figure 12).

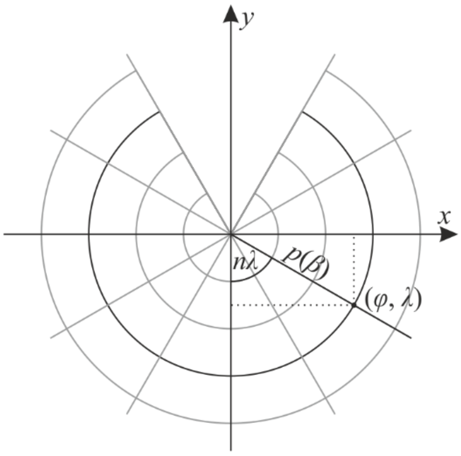

Figure 12. Graticule of normal aspect conic projections

The general projection equations of a normal aspect conic projection are therefore:

$$x = p(\beta)\sin(n\lambda)$$

$$y = -p(\beta)\cos(n\lambda)$$

Let's notice that a conic projection not necessarily projects the pole into a single point. If $p(0) > 0$, the pole appears as a circle arc.

Distortions along the graticule lines

The general formulas for the scale factors along the parallels (*h*) and the meridians (*k*) can be simplified after expressing the partial derivatives from the projection formulas above, knowing, that $\frac{d\beta}{d\varphi} = -1$, because $\beta = 90° - \varphi$.

$$\frac{\partial x}{\partial \varphi} = \frac{dp}{d\varphi}\sin(n\lambda) = \frac{dp}{d\beta}\frac{d\beta}{d\varphi}\sin(n\lambda) = -\frac{dp}{d\beta}\sin(n\lambda)$$

$$\frac{\partial y}{\partial \varphi} = -\frac{dp}{d\varphi}\cos(n\lambda) = \frac{dp}{d\beta}\cos(n\lambda)$$

$$\frac{\partial x}{\partial \lambda} = p(\beta) n \cos(n\lambda)$$

$$\frac{\partial y}{\partial \lambda} = p(\beta) n \sin(n\lambda)$$

therefore:

$$h = \frac{\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2}}{\cos\varphi} = \frac{p(\beta) n \sqrt{\cos^2(n\lambda) + \sin^2(n\lambda)}}{\cos\varphi} = \frac{p(\beta) n}{\cos\varphi} = \frac{p(\beta) n}{\sin\beta}$$

$$k = \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2} = \frac{dp}{d\beta}\sqrt{\sin^2(n\lambda) + \cos^2(n\lambda)} = \frac{dp}{d\beta}$$

Perspective Conic projection

The Perspective Conic projection can be constructed by placing a cone with its axis matching the sphere's axis of rotation, and projecting the surface of the sphere to the cone from a point along the axis (Figure 13). After unfolding the cone, the projected image will be similar to the one on Figure 12.

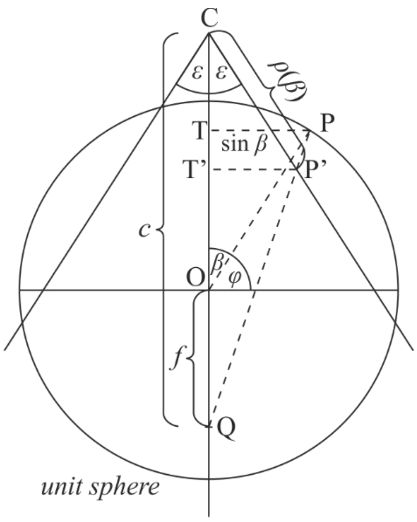

Figure 13. Perspective conic projection

The $p(\beta)$ radius for a given β polar distance can be calculated as follows: the triangles QTP and QT'P' are similar, so …

$$\frac{QT'}{QT} = \frac{T'P'}{TP}$$

where

$$TP = \sin\beta$$

$$T'P' = p(\beta)\sin\varepsilon$$

$$QT = f + \cos\beta$$

$$T'C = p(\beta)\cos\varepsilon$$

so

$$QT' = \frac{p(\beta)\sin\varepsilon\ (f + \cos\beta)}{\sin\beta}$$

$$c = QT' + T'C = \frac{p(\beta)[\sin\varepsilon\ (f + \cos\beta) + \cos\varepsilon\ \sin\beta]}{\sin\beta}$$

finally

$$p(\beta) = \frac{c\sin\beta}{\sin\varepsilon\ (f + \cos\beta) + \cos\varepsilon\ \sin\beta}$$

The Perspective Conic projection is not often used because its distortions are higher than the other conic projections.

Equidistant Conic projection

Let's construct a conic projection that has true scale meridians. This means that

$$k = \frac{dp}{d\beta} = 1$$

which implies

$$p(\beta) = \beta + p_0$$

where p_0 is a constant that can be chosen arbitrarily (along with the *n* angle proportion constant). As $p(\beta) = p_0$, if $\beta = 0$ (which is only at the North Pole) it means that p_0 is the radius of the polar arc.

In practical use the values of p_0 and *n* is not directly set but calculated from the position of the true scale latitudes of the projection. A conic projection can usually have two true scale parallels (sometimes also called "standard parallels"). Let's mark their polar distance by β_1 and β_2. As these are true scale parallels, their length on the sphere and on the map should be equal.

The length of β_1 parallel on the sphere is $2\pi\sin\beta_1$, while on the map it is $2\pi\ n\ p(\beta_1) = 2\pi\ n(\beta_1 + p_0)$. Therefore

$$\sin\beta_1 = n(\beta_1 + p_0)$$

and similarly

$$\sin\beta_2 = n(\beta_2 + p_0)$$

If we subtract the two above equations we get

$$\sin\beta_1 - \sin\beta_2 = n(\beta_1 - \beta_2)$$

and we can express *n*:

$$n = \frac{\sin\beta_1 - \sin\beta_2}{\beta_1 - \beta_2}$$

on the other hand,

$$n = \frac{\sin\beta_1}{\beta_1 + p_0} = \frac{\sin\beta_2}{\beta_2 + p_0}$$

so

$$\sin\beta_1 (\beta_2 + p_0) = \sin\beta_2 (\beta_1 + p_0)$$

from which p_0 also can be expressed:

$$p_0 = \frac{\beta_1 \sin\beta_2 - \beta_2 \sin\beta_1}{\sin\beta_1 - \sin\beta_2}$$

The Equidistant Conic projection is sometimes credited to *Joseph Nicolas De l'Isle* who extensively used it in his maps in the 18th Century. This projection is very often used for mapping regions that lie between the tropics and the polar circles.

Conformal Conic projection

In a conformal conic projection the scale factor along the meridian always equals the scale factor along the parallel ($h = k$):

$$k = \frac{dp}{d\beta} = \frac{p(\beta)n}{\sin\beta} = h$$

This is a separable differential equation that can be solved as follows:

$$\frac{1}{p(\beta)} dp = n \frac{1}{\sin\beta} d\beta$$

$$\int \frac{1}{p(\beta)} dp = n \int \frac{1}{\sin\beta} d\beta$$

$$\ln p(\beta) = n \ln\tan\frac{\beta}{2} + \ln d$$

where $\ln d$ is the constant of integration. Using the logarithmic identities

$$\ln p(\beta) = n \ln\tan\frac{\beta}{2} + \ln d = ln\left(d \tan^n \frac{\beta}{2}\right)$$

so

$$p(\beta) = d \tan^n \frac{\beta}{2}$$

To find the values of d and n, let's set the polar distance of the true scale parallels to β_1 and β_2. The length of the β_1 parallel is equal on the sphere and on the map:

$$2\pi \ \sin\beta_1 = 2\pi\, n\, p(\beta_1) = 2\pi\, n\, d \tan^n \frac{\beta_1}{2}$$

$$\sin\beta_1 = n\, d \tan^n \frac{\beta_1}{2}$$

similarly:

$$\sin\beta_2 = n\, d \tan^n \frac{\beta_2}{2}$$

If the two equations above are divided by each other:

$$\frac{\sin\beta_1}{\sin\beta_2} = \frac{\tan^n \frac{\beta_1}{2}}{\tan^n \frac{\beta_2}{2}}$$

Now let's use the logarithm of both sides:

$$\ln\frac{\sin\beta_1}{\sin\beta_2} = \ln\sin\beta_1 - \ln\sin\beta_2 = \ln\frac{\tan^n\frac{\beta_1}{2}}{\tan^n\frac{\beta_2}{2}} = n\left(\ln\tan\frac{\beta_1}{2} - \ln\tan\frac{\beta_2}{2}\right)$$

finally

$$n = \frac{\ln\sin\beta_1 - \ln\sin\beta_2}{\ln\tan\frac{\beta_1}{2} - \ln\tan\frac{\beta_2}{2}}$$

The value of d can now be expressed either from β_1 or from β_2:

$$d = \frac{\sin\beta_1}{n\tan^n\frac{\beta_1}{2}} = \frac{\sin\beta_2}{n\tan^n\frac{\beta_2}{2}}$$

This projection was developed by *Johann Heinrich Lambert* and is known as **Lambert Conformal Conic** or simply *LCC* projection. This projection is very often used in topographic mapping and for aeronautical charts.

Conic Equal-Area projection

To preserve areas $\tau = hk\sin\Theta = 1$ is required. As $\Theta = 90°$ therefore $\sin\Theta = 1$ for all the conic projections, this simplifies to $hk = 1$.

$h = \frac{p(\beta)n}{\sin\beta}$ and $k = \frac{dp}{d\beta}$, so

$$\frac{dp}{d\beta}\frac{p(\beta)n}{\sin\beta} = 1$$

this is again a separable differential equation; its solution:

$$\int p(\beta)\,dp = \frac{1}{n}\int \sin\beta\,d\beta$$

$$\frac{p(\beta)^2}{2} = \frac{1}{n}(-\cos\beta + d)$$

$$p(\beta)^2 = \frac{2}{n}(-\cos\beta + d)$$

where *d* is the constant of integration, and can be expressed as:

$$d = \frac{n[p(\beta)]^2}{2} + \cos\beta$$

Let's use the sign p_0 for the radius of the pole arc [$p_0 = p(0)$]. Then, if $\beta = 0$,

$$d = \frac{n[p(0)]^2}{2} + \cos 0 = \frac{n{p_0}^2}{2} + 1$$

This can be in the formula of $p(\beta)^2$:

$$p(\beta)^2 = \frac{2}{n}(-\cos\beta + d) = \frac{2}{n}\left(1 - \cos\beta + \frac{n{p_0}^2}{2}\right) = \frac{2}{n}\left(1 - \cos^2\frac{\beta}{2} + \sin^2\frac{\beta}{2} + \frac{n{p_0}^2}{2}\right)$$
$$= \frac{2}{n}\left(2\sin^2\frac{\beta}{2} + \frac{n{p_0}^2}{2}\right) = \frac{1}{n}\left(4\sin^2\frac{\beta}{2} + n{p_0}^2\right)$$

So

$$p(\beta) = \frac{1}{\sqrt{n}}\sqrt{4\sin^2\frac{\beta}{2} + n{p_0}^2}$$

To find the values of p_0 and n, let's set the polar distance of the true scale parallels to β_1 and β_2. The length of the β_1 parallel is equal on the sphere and on the map:

$$2\pi\ \sin\beta_1 = 2\pi\, n\, p(\beta_1) = 2\pi\ \frac{n}{\sqrt{n}}\sqrt{4\sin^2\frac{\beta_1}{2} + n{p_0}^2}$$

simplified:

$$\sin\beta_1 = \sqrt{n}\sqrt{4\sin^2\frac{\beta_1}{2} + n{p_0}^2}$$

$$4\sin^2\frac{\beta_1}{2} + n{p_0}^2 = \frac{\sin^2\beta_1}{n}$$

and similarly

$$4\sin^2\frac{\beta_2}{2} + n{p_0}^2 = \frac{\sin^2\beta_2}{n}$$

Subtracting these two equations we get

$$4\left(\sin^2\frac{\beta_1}{2} - \sin^2\frac{\beta_2}{2}\right) = \frac{\sin^2\beta_1 - \sin^2\beta_2}{n}$$

so

$$n = \frac{\sin^2\beta_1 - \sin^2\beta_2}{4\left(\sin^2\frac{\beta_1}{2} - \sin^2\frac{\beta_2}{2}\right)} = \frac{\sin^2\beta_1 - \sin^2\beta_2}{4\left(\frac{1-\cos\beta_1}{2} - \frac{1-\cos\beta_2}{2}\right)} = \frac{1 - \cos^2\beta_1 - 1 + \cos^2\beta_2}{2(\cos\beta_2 - \cos\beta_1)}$$
$$= \frac{\cos^2\beta_2 - \cos^2\beta_1}{2(\cos\beta_2 - \cos\beta_1)} = \frac{(\cos\beta_2 - \cos\beta_1)(\cos\beta_1 + \cos\beta_2)}{2(\cos\beta_2 - \cos\beta_1)} = \frac{\cos\beta_1 + \cos\beta_2}{2}$$

now we can express $n{p_0}^2$ as follows:

$$np_0{}^2 = \frac{\sin^2\beta_1}{n} - 4\sin^2\frac{\beta_1}{2} = \frac{\sin^2\beta_1 - 4n\sin^2\frac{\beta_1}{2}}{n} = \frac{4\sin^2\frac{\beta_1}{2}\cos^2\frac{\beta_1}{2} - 4\frac{\cos\beta_1+\cos\beta_2}{2}\sin^2\frac{\beta_1}{2}}{n}$$

$$= \frac{4\sin^2\frac{\beta_1}{2}}{n}\left(\cos^2\frac{\beta_1}{2} - \frac{\cos\beta_1+\cos\beta_2}{2}\right)$$

$$= \frac{4\sin^2\frac{\beta_1}{2}}{n}\left(\cos^2\frac{\beta_1}{2} - \frac{1-2\sin^2\frac{\beta_1}{2}+1-2\sin^2\frac{\beta_2}{2}}{2}\right)$$

$$= \frac{4\sin^2\frac{\beta_1}{2}}{n}\left(\cos^2\frac{\beta_1}{2} - \frac{2\cos^2\frac{\beta_1}{2}-2\sin^2\frac{\beta_2}{2}}{2}\right) = \frac{4\sin^2\frac{\beta_1}{2}\sin^2\frac{\beta_2}{2}}{n}$$

so finally

$$p(\beta) = \frac{1}{\sqrt{n}}\sqrt{4\sin^2\frac{\beta}{2} + np_0{}^2} = p(\beta) = \frac{1}{\sqrt{n}}\sqrt{4\sin^2\frac{\beta}{2} + \frac{4\sin^2\frac{\beta_1}{2}\sin^2\frac{\beta_2}{2}}{n}}$$

$$= 2\sqrt{\frac{\sin^2\frac{\beta}{2}}{n} + \frac{4\sin^2\frac{\beta_1}{2}\sin^2\frac{\beta_2}{2}}{n^2}}$$

The name of this equal area projection is “Albers Equal Area Conic” after the German mathematician Heinrich C. Albers who developed it in 1805. It is extensively used in Canada and in the United States.